139. On Characterizations of Spaces with G_δ-diagonals

By Takemi MIZOKAMI

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A space X is called to have a G_δ-diagonal if the diagonal Δ in $X\times X$ is a G_δ-set. A space X is called to have a regular G_δ-diagonal if Δ is a regular G_δ-set, that is, Δ is written by the following:

$$\Delta=\cap\{U_n/n\in N\}=\cap\{\overline{U}_n/n\in N\},$$

where U_n's are open sets containing Δ in $X\times X$ and N denotes the set of all natural numbers. Ceder in [1] characterized a G_δ-diagonal as follows:

Lemma 1. *A space X has a G_δ-diagonal iff (=if and only if) there is a sequence $\{\mathcal{U}_n/n\in N\}$ of open coverings of X such that for each point p in X*

$$p=\cap\{S(p,\mathcal{U}_n)/n\in N\}.$$

According to Zenor's result in [2], a regular G_δ-diagonal is characterized as follows:

Lemma 2. *A space X has a regular G_δ-diagonal iff there is a sequence $\{\mathcal{U}_n/n\in N\}$ of open coverings of X such that if p, q are distinct points in X, then there are an integer n and open sets U and V containing p and q, respectively, such that no member of $\mathcal{U}_n$ intersects both U and V.*

The object of the present paper is to characterize spaces with G_δ- or regular G_δ-diagonal by virtue of above lemmas as images of metric spaces under open mappings with some properties.

Theorem 1. *A space X has a G_δ-diagonal iff there is an open mapping (single-valued) f from a metric space T onto X such that*

$$d(f^{-1}(p),f^{-1}(q))>0 \text{ for distinct points } p,q\in X.$$

Proof. Only if part: Define T as follows:

$$T=\{(\alpha_1,\alpha_2,\cdots)\in N(A)/\cap\{U^n_{\alpha_n}/n\in N\}\neq\phi\},$$

where $\{\mathcal{U}_n=\{U^n_\alpha/\alpha\in A\}/n\in N\}$ is a sequence of open coverings of X satisfying the condition in Lemma 1. If we define a mapping $f: T\to X$ as follows;

$$f(\alpha)=\cap\{U^n_{\alpha_n}/n\in N\} \qquad \text{for } \alpha=(\alpha_1,\alpha_2,\cdots)\in T,$$

then f is clearly a single-valued mapping from T onto X. Since

$$f(N(\alpha_1,\cdots,\alpha_n))=\cap\{U^i_{\alpha_i}/1\leqq i\leqq n\},$$

it follows that f is open. Let p, q be distinct points in X; then by Lemma 1 we admit an integer n in N such that q does not belong to $S(p,\mathcal{U}_n)$. In this case it is proved that

$$d(f^{-1}(p), f^{-1}(q)) \geqq \frac{1}{n},$$

where d is a metric on a Baire's zero-dimensional metric space $N(A)$. Indeed, since

$$S_{1/n}(f^{-1}(p)) = \cup\{N(\alpha_1, \cdots, \alpha_n)/\alpha = (\alpha_1, \alpha_2, \cdots) \in f^{-1}(p)\},$$
$$q \overline{\in} f(S_{1/n}(f^{-1}(p))) = S(p, \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n).$$

This implies

$$S_{1/n}(f^{-1}(p)) \cap f^{-1}(q) = \phi.$$

Hence the distance between $f^{-1}(p)$ and $f^{-1}(q)$ is positive.

If part: Suppose T and f are given. Let $\{\mathcal{U}_n\}$ be a sequence of open coverings of T with mesh $\mathcal{U}_n < \frac{1}{n}$ such that $\{S(p, \mathcal{U}_n)/n \in N\}$ is a nbd (neighborhood) basis of each point p in T. If we set

$$\mathcal{V}_n = f(\mathcal{U}_n) = \{f(U)/U \in \mathcal{U}_n\},$$

then $\{\mathcal{V}_n\}$ is the desired sequence. Indeed, each $\mathcal{V}_n$ is an open covering of X because f is open and onto. Assume that p, q are distinct points in X. Then there is an integer n in N such that

$$d(f^{-1}(p), f^{-1}(q)) \geqq \frac{1}{n},$$

which implies

$$S_{1/n}(f^{-1}(p)) \cap f^{-1}(q) = \phi,$$

and consequently we have

$$q \overline{\in} f(S_{1/n}(f^{-1}(p))).$$

Since each mech $\mathcal{U}_n < \frac{1}{n}$, it follows that

$$q \overline{\in} S(p, \mathcal{V}_n) = f(S(f^{-1}(p), \mathcal{V}_n)).$$

Hence by Lemma 1 Δ is G_δ. Thus the proof is completed.

Theorem 2. *A space X has a regular G_δ-diagonal iff there is an open mapping f from a metric space T onto X such that for any pair of distinct points p, q in X, there exist nbds U and V of p and q, respectively, such that*

$$d(f^{-1}(U), f^{-1}(V)) > 0.$$

Proof. Only if part: The construction of T and f is similar to that of Theorem 1, except the fact that a sequence $\{\mathcal{U}_n/n \in N\}$ satisfies the condition in Lemma 2 in place of Lemma 1. Then it is trivial that f is open and onto. Suppose we are given a pair of distinct points p, q in X. Then we get an integer n in N and nbds U and V of p and q, respectively, such that no member of $\mathcal{U}_n$ intersects both U and V, that is, $S(U, \mathcal{U}_n) \cap V = \phi$. Observe

$$f(S_{1/n}(f^{-1}(U))) = S(U, \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n).$$

Thus we have

$$S_{1/n}(f^{-1}(U)) \cap f^{-1}(V) = \phi,$$

implying

$$d(f^{-1}(U), f^{-1}(V)) > 0.$$

If part: Construct a sequence $\{\mathcal{U}_n / n \in N\}$ of open coverings of X in the same fashion as in the proof of Theorem 1. Suppose p and q are distinct points in X. Then by assumption on f we obtain nbd U and V of p and q, respectively, such that

$$d(f^{-1}(U)), f^{-1}(V)) \geqq \frac{1}{n} \qquad \text{for some } n \in N.$$

This implies

$$S_{1/n}(f^{-1}(U)) \cap f^{-1}(V) = \phi,$$

which implies

$$S(U, \mathcal{V}_n) \cap V = \phi,$$

proving that $\{\mathcal{V}_n\}$ is a sequence in Lemma 2. Hence X has a regular G_δ-diagonal. Thus the proof is completed.

Hodel in [3] introduced the notion of G_δ^*-diagonal as follows: *A* space X is called to have a G_δ^*-diagonal if there is a sequence $\{\mathcal{U}_n / n \in N\}$ of open coverings of X such that if for any pair of distinct points p, q in X there is an integer n in N such that p does not belong to the closure of $S(q, \mathcal{U}_n)$. Such a sequence is called a G_δ^*-sequence for X. It is to be noted that a G_δ^*-diagonal implies a G_δ-diagonal and that a regular G_δ-diagonal implies a G_δ^*-diagonal.

Theorem 3. *A space X has a G_δ^*-diagonal iff there is an open mapping f from a metric space T onto X such that for any pair of distinct points p, q in X thre is a nbd V of p satisfying*

$$d(f^{-1}(V), f^{-1}(q)) > 0.$$

The proof is similar to that of Theorem 1.

According to Heath in [4] a space X is called to have a G_δ-diagonal with 3-link property if there is a sequence $\{\mathcal{U}_n / n \in N\}$ of open coverings of X such that if p and q are distinct points in X, then there is an integer n in N such that no member of $\mathcal{U}_n$ intersects both $S(p, \mathcal{U}_n)$ and $S(q, \mathcal{U}_n)$. Which respect to this G_δ-diagonal we have a comparable characterization as follows:

Theorem 4. *A space X has a G_δ-diagonal with 3-link property iff there is an open mapping f from a metric space T onto X such that for any pair of distinct points p and q in X and for some n in N*

$$d(f^{-1}(f(S_{1/n}(f^{-1}(p)))), f^{-1}(f(S_{1/n}(f^{-1}(q))))) > 0.$$

Proof. Only if part: For a given sequence $\{\mathcal{U}_n\}$ of open coverings, we construct T and f in the same fashion as seen in the proof of Theorem 1. Let p, q be distinct points in X. Then we have an integer n in N such that $q \overline{\in} S^3(p, \mathcal{U}_n)$. Observe that

$$S^3(p, \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_{1/n}) = f(S_n(f^{-1}(f(S_{1/n}(f^{-1}f(S_{1/n}(f^{-1}(p)))))))).$$

Since

$$S^3(p, \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n) \subset S^3(p, \mathcal{U}_n),$$

we obtain

$$q \overline{\in} f(S_{1/n}(f^{-1}(f(S_{1/n}(f^{-1}(f(S_{1/n}(f^{-1}(p))))))))),$$

from which we conclude that

$$d(f^{-1}(f(S_{1/n}(f^{-1}(p)))), f^{-1}(f(S_{1/n}(f^{-1}(q))))) > 0.$$

If part: We construct a sequence $\{\mathcal{V}_n / n \in N\}$ of open coverings of X by the same way as in the proof of Theorem 1. Then we can show by using the property of f that $\{\mathcal{V}_n\}$ satisfies the 3-link property, and hence the proof is completed.

References

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