127. On Schwarz's Lemma for $\Delta u + c(x)u = 0$

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1. **Introduction.** The famous Schwarz's lemma in the complex function theory states that if $f(z)$ is holomorphic in $|z|<1$, and if $f(0)=0$ (or $|f(z)|\le \text{const.}\,|z|$), then the estimate: $|f(z)|\le |z|\max_{|\zeta|=1}|f(\zeta)|$ holds for $|z|<1$. Many theorems in the complex function theory have been generalized with great success to the case of harmonic functions, or more generally, solutions of the second order elliptic differential equations with variable coefficients. The Schwarz's lemma, however, does not seem to have been generalized previously even to the case of harmonic functions. The main purpose of the present paper is to generalize this lemma to solutions of the equations of the form: $\Delta u + c(x)u=0$. As corollaries of the generalized Schwarz's lemma, we can obtain the generalizations of the Hadamard three-circles theorem [which states; if $f(z)$ is holomorphic in $|z|<R$, then $\log \max_{\theta}|f(re^{i\theta})|$ is a convex function of $\log r (0<r<R)$], and the Liouville's theorem [which states; if an entire function $f(z)$ satisfies $0(|z|^k)$ as $|z|\to\infty$ (k; non-negative integer), then $f(z)$ is a polynomial of at most degree k]. The extension of the results below to the case of the general second order elliptic equations with variable coefficients will be published elsewhere.

2. **Notations.** R^n denotes the n-dimensional real Euclidean space, and C^n the n-dimensional complex Euclidean space. We denote the inner product and the norm in C^n (R^n) by $\langle\cdot,\cdot\rangle$ and $|\cdot|$. We set $S_R=\{x\in R^n;\, 0<|x|<R\}$. Let $L^2(\Sigma)$ be the L^2-space of C^m-valued functions defined on the unit surface $\Sigma=\{x\in R^n;\,|x|=1\}$. Then $L^2(\Sigma)$ is the Hilbert space with the usual inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$: $(u,v)=\int_\Sigma \langle u(\xi), v(\xi)\rangle d\sigma_\xi$ and $\|u\|=(u,u)^{1/2}$ where $d\sigma_\xi$ denotes the surface area element on the unit surface Σ. $H^2(\Sigma)$ denotes the set of all functions in $L^2(\Sigma)$ whose distribution derivatives up to order 2 belong to $L^2(\Sigma)$. Now putting $u(r,\xi)=u(x)$ ($r=|x|$; $\zeta=x/|x|$), we shall then regard a function $u(x)$ defined in the sphere $|x|<R$ as an $L^2(\Sigma)$-valued function of r, and simply write $u(r)$ for $u(r,\xi)$ (or $u(x)$). Finally, for multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $x=(x_1,\cdots,x_n)\in R^n$, we set $x^\alpha=x_1^{\alpha_1}\cdots \alpha_n^{\alpha_n}$ and $D^\alpha=\partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$.

3. Results. Let us consider the equation of the form

$$\Delta u + \boldsymbol{c}(x)u = 0 \qquad (x \in S_R) \tag{1}$$

where $\Delta = (\partial^2/\partial x_1^2) + \cdots + (\partial^2/\partial x_n^2)$, $u(x)$ is an m-vector function of x and $\boldsymbol{c}(x)$ is an $(m \times m)$ matrix-valued function of x. Concerning $\boldsymbol{c}(x)$, we shall make the following assumption throughout the present paper.

Assumption on $\boldsymbol{c}(x)$. *The $(m \times m)$ matrix-valued function $\boldsymbol{c}(x)$ is hermitean for each fixed x in S_R, is continuously differentiable with respect to $x \in S_R$, and satisfies the conditions*:

$$\langle \boldsymbol{c}(x)\zeta, \zeta \rangle \leq 0 \qquad (x \in S_R, \zeta \in C^m) \tag{2}$$

and

$$\left\langle \left(|x| \frac{\partial}{\partial |x|} c(x) + 2\boldsymbol{c}(x) \right) \zeta, \zeta \right\rangle \leq 0 \qquad (x \in S_R, \zeta \in C^m) \tag{3}$$

Now we can state the results of the present paper.

Theorem 1 (Schwarz's lemma). *Let the above assumption on* $\boldsymbol{c}$ *be satisfied. Let u be a solution of equation* (1). *If u satisfies the estimate*:

$$\|u(r)\| \leq M r^{\beta} \qquad (0 < r < R) \tag{4}$$

where M is some constant and β is some real number with $-(n-2)/2 \leq \beta$, then we have

$$\|u(r)\| \leq \left(\frac{r}{r_0} \right)^{\beta} \|u(r_0)\| \tag{5}$$

for all r, r_0 with $0 < r < r_0 < R$.

Theorem 2 (Hadamard's three-spheres theorem). *Let the above assumption on* $\boldsymbol{c}$ *be satisfied. Let u be a solution of* (1). *Then* $\log \|u(r)\|$ *is a convex function of* $\log r$, *that is,*

$$\log \|u(r_2)\| \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log \|u(r_1)\| + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log \|u(r_3)\| \tag{6}$$

for $0 < r_1 < r_2 < r_3 < R$.

This theorem was generalized to the case of general second order elliptic equations by E. Landis [3], [4], S. Agmon [1], K. Miller [5] and Protter-Weinberger [6], in various forms.

Theorem 3 (Liouville's theorem). *Let $R = \infty$. Let the above assumption on* $\boldsymbol{c}$ *be satisfied. Let u be a k-times continuously differentiable solution of* (1) *in R^n. If u satisfies $|u(x)| = o(|x|^{k+1})$ as $|x| \to \infty$, then u is uniquely determined by the values $D^{\alpha}u(o)$ $(|\alpha| \leq k)$ of $u(x)$ at the origin. In particular, if u is harmonic in R^n, i.e., $\Delta u = 0$, and if $|u(x)| = o(|x|^{k+1})$ as $|x| \to \infty$, then u is a polynomial of at most degree k.*

This theorem with $k = 0$ and with $o(|x|)$ replaced by $0(1)$ has been extended to the general second order elliptic differential equations (see, e.g., Gilbarg-Serrin [2]), while for general k, the theorem does not seem to the writer to have been shown previously.

4. Proof of Theorem 1. Using the polar coordinate (r, ξ); $r=|x|$, $\xi=x/|x|$, we shall rewrite equation (1) in the form

$$u_{rr}(r)+\frac{n-1}{r}u_r(r)+\frac{\Lambda}{r^2}u(r)+c(r,\xi)u(r)=0 \tag{7}$$

where $u_r(r)=(d/dr)u(r)$, $c(r,\xi)=c(x)$, and Λ denotes the Laplace-Beltrami operator on the unit sphere Σ. We note here that if we define the domain of $\Lambda=H^2(\Sigma)$, then Λ becomes the non-positive self-adjoint operator in $L^2(\Sigma)$. Changing the variable $r\to t=1/r$, and putting $v(t,\xi)=t^{-\alpha}u(t^{-1},\xi)$ $(\alpha=(n-2)/2)$, we see that the v satisfies

$$L[v]\equiv v_{tt}+\frac{1}{t}v_t+A(t)v=0, \qquad t>T\ (T=1/R) \tag{8}$$

where

$$A(t)=\frac{1}{t^2}\Lambda-\frac{(n-2)^2}{4t^2}+\frac{1}{t^4}c(t^{-1},\xi).$$

Let us define the domain $D(A(t))$ of $A(t)$ by $D(A(t))=H^2(\Sigma)$. Then we have:

Lemma 1. *We have*

(i) *for each $t>T$, $A(t)$ is a closed symmetric operator in $L^2(\Sigma)$ with*

$$(A(t)y,y)\le 0 \qquad (\text{for all } y \text{ in } D(A(t))) \tag{9}$$

(ii) *the estimate*

$$((d/dt)A(t)y,y)\ge -\frac{2}{t}(A(t)y,y) \tag{10}$$

holds for all $t>T$ and y in $D(A(t))$.

Proof. Since Λ is non-positive and symmetric, and since $\langle c\zeta,\zeta\rangle\le 0$, (i) easily follows. Differentiating $A(t)y$ $(y\in H^2(\Sigma))$ with respect to t, we have $(d/dt)A(t)y=-2t^{-2}A(t)y+t^{-5}\{t(\partial/\partial t)c(t^{-1},\xi)-2c(t^{-1},\xi)\}y$. Since, by the assumption on c, $\langle t(\partial/\partial t)c(t^{-1},\xi)y-2c(t^{-1},\xi)y, y\rangle\ge 0$, we have $\langle (d/dt)A(t)y,y\rangle\ge -\frac{2}{t}\langle A(t)y,y\rangle$. Integrating with respect to ξ over Σ, we have (10). This completes the proof of the lemma.

We now define

$$m(t)=\|v_t\|^2+(A(t)v,v).$$

Then we have the following two cases: (I) $m(t_0)>0$ for some $t_0>T$; (II) $m(t)\le 0$ for all $t>T$. (a) the case (I). Let t_1 be the supremum for all t such that $m(s)>0$ for all s with $t_0\le s<t$. By the continuity in t of $m(t)$, we then have $t_0<t_1$. Writing $(d/dt)A(t)y=\dot A(t)y$, $(d/dt)v=v'$, and differentiating $m(t)$ in t, we find

$$\begin{aligned} m(t)'&=2\,\mathrm{Re}\,(v'',v')+(\dot A(t)v,v)+2\,\mathrm{Re}\,(A(t)v,v') \\ &=2\,\mathrm{Re}\,(L[v],v')-\frac{2}{t}\|v'\|^2+(\dot A(t)v,v) \\ &\ge -\frac{2}{t}\{\|v'\|^2+(A(t)v,v\}=-\frac{2}{t}m(t) \qquad \text{[by (8) and (10)].} \end{aligned}$$

Hence $(t^2m(t))'\geq 0$, and so, by the integration,

(11) $$m(t)\geq (t_0/t)^2 m(t_0), \qquad (t_0<t<t_1).$$

If $t_1<\infty$, then we have $m(t_1)>0$ and hence $m(t)>0$ for some interval $[t_1, t_1+\delta)$ $(\delta>0)$: a contradiction to the definition of t_1. Hence t_1 must be the infinity, and (11) holds for all $t>t_0$. Now, by direct calculation,

$$\begin{aligned}(d^2/ds^2)(v,v) &= 2\,\mathrm{Re}\,(L[v],v)-\frac{2}{s}\,\mathrm{Re}\,(v',v)-2(A(s)v,v)+2\,\|v'\|^2\\ &= -\frac{2}{s}\,\mathrm{Re}\,(v',v)+2m(s)-4(A(s)v,v) \qquad (12)\\ &\geq -\frac{2}{s}\,\mathrm{Re}\,(v',v)+2(t_0/s)^2 m(t_0) \qquad \text{[by (9) and (11)]}.\end{aligned}$$

Multiplying both sides of (12) by $(t-s)(s-t_0)$, and integrating with respect to s over the interval (t_0, t), we have

$$\begin{aligned}&(t-t_0)\,\|v(t)\|^2+(t-t_0)\,\|v(t_0)\|^2-2\int_{t_0}^{t}\|v(s)\|^2 ds\\ &=\int_{t_0}^{t}(t-s)(s-t_0)\frac{d^2}{ds^2}\,\|v(s)\|^2 ds\\ &\geq -\int_{t_0}^{t}\left(2-\frac{t_0 t}{s^2}\right)\|v(s)\|^2 ds+2(t-t_0)t_0^2 m(t_0)[\log\,(t/t_0)-2]\end{aligned}$$

from which it follows that

$$(t-t_0)\,\|v(t)\|^2+(t-t_0)\,\|v(t_0)\|^2\geq 2(t-t_0)t_0^2 m(t_0)[\log\,(t/t_0)-2].$$

Hence, dividing both sides by $t-t_0$, and expressing this inequality in terms of the original variable r and the function $u(r)$ $(=r^{-\alpha}v(r^{-1}))$, we get

(13) $$\|u(r)\|^2+(r_0/r)^{2\alpha}\,\|u(r_0)\|^2\geq 2r^{-2\alpha}r_0^{-2}m(t_0)[\log\,(r_0/r)-2]$$

where $r_0=1/t_0$. Since by the assumption, $\|u(r)\|^2=O(r^{-n+2})$ as $r\to 0$, we see that (13) is impossible. Hence our assumption excludes the possibility of the case (I). (b) the case (II). Let t_0 be any number with $v(t_0)\neq 0$ $(t>T)$. We set $t_1=\sup\,\{t;\, v(s)\neq 0 \text{ for all } s \text{ with } t_0\leq s<t\}$. Setting $\ell(t)=\log\,\|v(t)\|^2$, we differentiate $\ell(t)$ in t. The results are:

(14) $$\ell'(t)=2\,\mathrm{Re}\,(v'(t),v(t))/\|v(t)\|^2.$$

(15) $$\ell''(t)=\frac{2}{\|v(t)\|^2}\{Re\,(v''(t),v(t))+\|v'(t)\|^2-2[\mathrm{Re}\,(v'(t),v(t))]^2/\|v(t)\|^2\}.$$

By (8), $\mathrm{Re}\,(v'',v)=-t^{-1}\,\mathrm{Re}\,(v',v)-(A(t)v,v)$. On the other hand, the condition that $m(t)\leq 0$ implies that $-(A(t)v,v)\geq\|v'\|^2$. Hence, $\mathrm{Re}\,(v'',v)\geq\|v'\|^2-t^{-1}\,\mathrm{Re}\,(v',v)$. Hence, we have

$$\begin{aligned}\ell''(t) &\geq \frac{2}{\|v\|^2}\{2\,\|v'\|^2-t^{-1}\,\mathrm{Re}\,(v',v)-2[\mathrm{Re}\,(v',v)]^2/\|v\|^2\} \qquad (16)\\ &\geq -t^{-1}\ell'(t) \qquad \text{[by (14) and the Schwarz inequality]}\end{aligned}$$

and so $(t\ell'(t))'\geq 0$. Integrating with respect to t, we set

(17) $$\ell(t)\geq \ell(t_0)+t_0\ell'(t_0)\log\,(t/t_0), \qquad t_0<t<t_1.$$

This implies that if $t_1<\infty$, then $\ell(t_1)\neq -\infty$, and so $v(t_1)\neq 0$. Hence

$v(t)\neq 0$ for some interval $[t_1, t_1+\delta)$ (δ; some positive constant); a contradiction to the definition of t_1. Hence t_1 must be the infinity. This implies: if $v(t_0)\neq 0$, then $v(t)\neq 0$ for $t>t_0$ and (17) holds for all $t>t_0$. If $\|u(r)\|\leq Mr^{\beta}(-(n-2)/2\leq\beta, M$; positive constant), and so if $\|v(t)\|\leq Mt^{-\alpha-\beta}$, then we have $\ell(t)\leq 2\log M-2(\alpha+\beta)\log t$. By (16), $\ell(t_0)+t_0\ell'(t_0)\log(t/t_0)\leq 2\log M-2(\alpha+\beta)\log t$. Dividing both sides by $\log t$, and letting $t\to\infty$, we obtain $t_0\ell'(t_0)\leq -2\alpha-2\beta$, which holds for any t_0 with $v(t_0)\neq 0$. If we fix a $t_0(>T)$ such that $v(t_0)\neq 0$, then we have the inequality $t\ell'(t)\leq -2\alpha-2\beta$, $t>t_0$, since $v(t)\neq 0$ for all $t>t_0$. Integrating in t, we have

$$\log\|v(t)\|^2-\log\|v(t_0)\|^2\leq -2(\alpha+\beta)(\log t-\log t_0),$$

from which it follows that $\|v(t)\|\leq (t_0/t)^{\alpha+\beta}\|v(t_0)\|$. Let us express the results so far obtained in terms of the original variable r $(=t^{-1})$ and the original function $u(r)$ $(=r^{-\alpha}v(r^{-1}))$. Then if $u(r_0)\neq 0$, then $u(r)\neq 0$ for $0<r<r_0$ and the estimate

$$\|u(r)\|\leq (r/r_0)^{\beta}\|u(r_0)\|$$

holds. Now if it is shown that the solution u satisfies either $u\equiv 0$, or $u(r)\neq 0$ for all r in $(0, R)$, then the proof will be completed. To see this, we have only to show the following lemma.

Lemma 2. *Let be a solution of* (1). *If* $u(r_0)\neq 0$ *for some* r_0 $(0<r_0<R)$, *then* $u(r)\neq 0$ *for* $r_0<r<R$.

Proof. We define $v(t)$ by $v(t)=t^{-\alpha}u(1/t)$ $(\alpha=(n-2)/2)$. We first show that $\|v(t)\|$ is decreasing in t. As in (12), we have

$$(d^2/dt^2)\|v(t)\|^2=-2t^{-1}\operatorname{Re}(v', v)-2(A(t)v, v)+2\|v'\|^2$$
$$\geq -2t^{-1}\operatorname{Re}(v'v) \qquad [\text{by } (9)]$$

and so $(t(\|v(t)\|^2)')'\geq 0$. Integrating this inequality with respect to t over (t_1, t_2), we obtain

$$\|v(t_2)\|^2\geq\|v(t_1)\|^2+[s(d/ds)\|v(s)\|^2]_{s=t_1}\log(t_2/t_1) \tag{18}$$

for $T<t_1<t_2$ $(T=R^{-1})$. If $(d/ds)\|v(t_1)\|^2>0$ for some t_1 then inequality (18) shows that $\|v(t_2)\|\to\infty$ for $t_2\to\infty$, which contradicts the assumption that $\|v(t)\|=t^{-\alpha}\|u(t^{-1})\|=0(1)$ as $t\to\infty$. Hence $(d/dt)\|v(t)\|^2\leq 0$ for all $t>T$, showing that $\|v(t)\|$ is decreasing in t. We now suppose that $u(r^*)=0$ for some $r_0<r^*<R$. Then $v(t^*)=0$ $(t^*=1/r^*)$. Since $t^*<r_0^{-1}$, and since $\|v(t)\|$ is decreasing in t, we see $\|v(r_0^{-1})\|=0$, contradicting the assumption $u(r_0)\neq 0$. This proves the lemma.

5. Proof of Theorem 2. Setting $k(t)=\log\|u(t^{-1})\|^2$, and noting that $k(t)=2\alpha\log t+\ell(t)$, we have, by (16),

$$k''(t)\geq -t^{-1}k'(t).$$

Changing the variable $t\to s=-\log t$ and noting that $(d/dt)=-t^{-1}(d/ds)$, $(d^2/dt^2)=t^{-2}(d^2/ds^2+d/ds)$, we have $s^2(d^2/ds^2)\log\|u(e^s)\|^2\geq 0$, and so $(d^2/ds^2)\log\|u(e^s)\|^2\geq 0$, which implies $\log\|u(r)\|$ is a convex function of $s=\log r$.

6. Proof of Theorem 3. Let u_1 be any solution of (1) such that $u_1(x)=o(|x|^{k+1})$ as $|x|\to\infty$, and such that $D^\alpha u_1(0)=D^\alpha u(0)$ $(|\alpha|\leq k)$. Then $w=u-u_1$ is also a solution of (1) such that $w(x)=o(|x|^{k+1})$ as $|x|\to\infty$, and such that $D^\alpha w(0)=0$ $(|\alpha|\leq k)$. Hence $|w(x)|\leq \text{const.}\ |x|^{k+1}$. Applying Schwarz's lemma, we have $\|w(r)\|\leq (r/r_0)^{k+1}\|w(r_0)\|$ for all r, r_0 with $0<r<r_0$. Since $|w(x)|=o(r^{k+1})$, the right hand side $(r/r_0)^{k+1}\|w(r_0)\|$ tends to zero as $r_0\to\infty$. Hence $w(r)=0$ for all $r>0$, proving the first part of the theorem. Let u be harmonic in R^n. We expand $u(x)$ at the origin in the form: $u(x)=u_1(x)+u_2(x)$ where $u_1(x)=\sum_{|\alpha|\leq k}\frac{1}{\alpha!}D^\alpha u(0)\cdot x^\alpha$. Then it is easy to see that u_1 is harmonic in R^n, and that $u_1(x)=O(|x|^k)$ as $|x|\to\infty$, $D^\alpha u_1(0)=D^\alpha u(0)$ $(|\alpha|\leq k)$. Hence it follows from the first part of the theorem just proved that $u(x)=u_1(x)$. This completes the proof of the theorem.

References

[1] Agmon, S.: Unicité et convexité dans les problèms differentielles. Sem. d'Analyse Sup., Univ. de Montreal (1965).
[2] Gilbarg, D., and J. Serrin: On isolated singularities of solutions of second order elliptic differential equations. J. d'Analyse Math., **4**, 309–336 (1954–56).
[3] Landis, E.: A three-sphere theorem. Dokl. Akas. Nauk S.S.S.R., **148**, 277–279 (1963).
[4] ——: Some problems of the qualitative theorey of second order elliptic equations (case of several variables). Uspekhi Mat. Nauk, **18**, 3–62 (1963). Translated in Russian Math. Surveys, **18**, 1–62 (1963).
[5] Miller, K.: Three circle theorems in parial differential equations and applications to improperly posed problems. Arch. for Rat. Mech. and anal., **16**, 126–154 (1964).
[6] Protter, M., and H. Weingerger: Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs, N. J. (1967).
[7] Vyborny, R.: The Hadamard three-circles theorems for partial differential equations. Bull. Amer. Math. Soc., **80**, 81–84 (1974).