122. *The Fixed Point Set of an Involution and Theorems of the Borsuk-Ulam Type*

By Akio HATTORI

Department of Mathematics, University of Tokyo

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1. Statement of results. In this note, h^* will denote either the unoriented cobordism theory $\mathfrak{N}^*$ or the usual cohomology theory with $\boldsymbol{Z}_2$-coefficients $H^*(\ ;\boldsymbol{Z}_2)$. The corresponding equivariant cohomology theory for $\boldsymbol{Z}_2$-spaces will be denoted by $h^*_{\boldsymbol{Z}_2}$.

Let M be a manifold and σ an involution on M.[1] We define an embedding $\Delta: M \to M^2 = M\times M$ by $\Delta(x)=(x,\sigma x)$. Then Δ is equivariant with respect to the involution σ on M and the involution T on M^2 which is defined by $T(x_1,x_2)=(x_2,x_1)$. Let $\Delta_!: h^q_{\boldsymbol{Z}_2}(M)\to h^{q+m}_{\boldsymbol{Z}_2}(M^2)$ denote the Gysin homomorphism for Δ, where $m=\dim M$. We put $\theta(\sigma)=\Delta_!(1) \in h^m_{\boldsymbol{Z}_2}(M^2)$.

In the present note we shall give an explicit formula for $\theta(\sigma)$ and apply it to get theorems of the Borsuk-Ulam type. Our results generalize those of Nakaoka [3], [4]. From the formula for $\theta(\sigma)$ we shall also derive a sort of integrality theorem concernining the fixed point set of σ; see Theorem 4. Detailed accounts will appear elsewhere.

Let S^∞ be the infinite dimensional sphere with the antipodal involution. The projection $\pi: S^\infty\times M^2 \to S^\infty \times_{\boldsymbol{Z}_2} M^2$ induces the Gysin homomorphism $\pi_!: h^*(M^2)\to h^*_{\boldsymbol{Z}_2}(M^2)$ and the usual homomorphism $\pi^*: h^*_{\boldsymbol{Z}_2}(M^2) \to h^*(M^2)$. Let $d: M\to M^2$ be the diagonal map. Since $d(M)$ is the fixed point set of T, $h^*_{\boldsymbol{Z}_2}(d(M))$ is isomorphic to $h^*_{\boldsymbol{Z}_2}(pt)\otimes_{h^*(pt)} h^*(M)$ and d induces $d^*: h^*_{\boldsymbol{Z}_2}(M^2)\to h^*_{\boldsymbol{Z}_2}(pt)\otimes_{h^*(pt)} h^*(M)$.

Lemma 1. *The homomorphism*

$$\pi^*\oplus d^*: h^*_{\boldsymbol{Z}_2}(M^2)\to h^*(M^2)\oplus(h^*_{\boldsymbol{Z}_2}(pt)\otimes_{h^*(pt)} h^*(M))$$

is injective.

We denote by S the multiplicative set $\{w_1^k \mid k\geq 1\}$ in $h^*_{\boldsymbol{Z}_2}(pt)=h^*(P^\infty)$ where w_1 is the universal first Stiefel-Whitney class. If X is a $\boldsymbol{Z}_2$-space then $h^*_{\boldsymbol{Z}_2}(X)$ is an $h^*_{\boldsymbol{Z}_2}(pt)$-module and we can consider the localized ring $S^{-1}h^*_{\boldsymbol{Z}_2}(X)$ of $h^*_{\boldsymbol{Z}_2}(X)$ with respect to S. Note that $h^*_{\boldsymbol{Z}_2}(pt)$ is isomorphic to a formal power series ring $h^*(pt)[[w_1]]$ and $h^*_{\boldsymbol{Z}_2}(pt)\otimes_{h^*(pt)} h^*(M)$

1) In this note we work in the smooth category. All manifolds will be connected, compact and without boundary unless otherwise stated.

is canonically embedded in $(S^{-1}h^*_{Z_2}(pt)) \underset{h^*(pt)}{\otimes} h^*(M)$.

To state our main theorem we need some notations. Let $P: h^q(M) \to h^{2q}_{Z_2}(M^2)$ be the Steenrod-tom Dieck operation; see [4], [6]. For $u \in h^q(M)$ we define $P_0(u)$ to be $d^*P(u)/w_1^{2q}$. Then P_0 is extended to a ring homomorphism $P_0: h^*(M) \to (S^{-1}h^*_{Z_2}(pt)) \underset{h^*(pt)}{\otimes} h^*(M)$. For a real vector bundle ξ over a CW-complex X its h^*-theory Wu classes $v_\alpha(\xi) \in h^*(X)$ are defined in a similar way as in [5]. The Wu classes of the tangent bundle of a manifold X will be denoted by $v_\alpha(X)$. Finally we define $a_j(x) \in h^*(pt)[[x]]$ by

$$F(x, y) = \sum_{0 \leq j} a_j(x) y^j$$

where F is the formal group law of the theory h^*. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots)$ we put $a^\alpha(x) = \prod_{1 \leq j} a_j^{\alpha_j}(x)$, $l(\alpha) = \sum_j \alpha_j$ and $|\alpha| = \sum_j j\alpha_j$, cf. [6].

Theorem 2. *Let M be a manifold and σ an involution on M. Let F be the fixed point set of σ. F is a disjoint union of submanifolds $F_1, \cdots, F_l$.*

i) *$\pi^*\theta(\sigma) \in h^*(M^2)$ is given by*

$$\pi^*\theta(\sigma) = \Delta_!(1)$$

where the $\Delta_!$ on the right-hand side is the usual Gysin homomorphism $h^*(M) \to h^*(M^2)$. If $\{u_i\}$ is a homogeneous $h^*(pt)$ basis of $h^*(M)$ and $\Delta_!(1) = \sum a_{ij} u_i \times u_j$ with $a_{ij} \in h^*(pt)$ then the a_{ij}'s satisfy the relation

$$\sum_j a_{ij} c_{jk} = \delta_{ik} \qquad (\textit{the Kronecker } \delta)$$

where $c_{jk} = p_!(u_j \cup \sigma^* u_k)$ with $p: M \to pt$.

ii) $d^*\theta(\sigma) \in h^*_{Z_2}(pt) \underset{h^*(pt)}{\otimes} h^*(M) \subset (S^{-1}h^*_{Z_2}(pt)) \underset{h^*(pt)}{\otimes} h^*(M)$

is given by

$$d^*\theta(\sigma) = w_1^m \frac{\sum_{i=1}^{l} \sum_\alpha w_1^{2(-l(\alpha)+|\alpha|)} a^{2\alpha}(w_1) P_0(j_!(v_\alpha(F_i)^2))}{\sum_\alpha w_1^{-l(\alpha)+|\alpha|} a^\alpha(w_1) P_0(v_\alpha(M))}$$

where $j_!$ is the Gysin homomorphism of the inclusion $j: F \subset M$ and $m = \dim M$.

Remark 3. *In Theorem 2, when the theory h^* is the usual cohomology theory $H^*(\ ; Z_2)$, the formula for $d^*\theta(\sigma)$ reduces to*

$$d^*\theta(\sigma) = w_1^m P_0\left(\left\{\sum_{i=1}^{l} \sum_{s=0}^{[f_i/2]} j_!(v_s(F_i)^2)\right\} \Big/ \left\{\sum_{s=0}^{[m/2]} v_s(M)\right\}\right)$$

where $f_i = \dim F_i$.

Theorem 4. *Let M, σ and F_i be as in Theorem 2. Suppose that $h^* = H^*(\ ; Z_2)$. If we write*

$$\sum_{i=1}^{l} \sum_{s=0}^{[f_i/2]} j_!(v_s(F_i)^2) \Big/ \sum_{s=0}^{[m/2]} v_s(M) = \sum_{i=0}^{m} u_i$$

where $u_i \in H^i(M; Z_2)$, then we must have

$$u_i=0 \qquad \text{for } i>\frac{m}{2}.$$

Corollary 5. *Under the situation of Theorem 4 the element $\theta(\sigma)\in H^m_{Z_2}(M^2;Z_2)$ is given by*

$$\theta(\sigma)=\sum_{i=0}^{[m/2]} w_1^{m-2i}P(u_i)+\theta_1$$

where θ_1 is characterized by the conditions

a) $\theta_1\in\pi_!$*-image*

and

b) $\pi^*\theta_1=\Delta_!(1)+u_{m/2}\times u_{m/2}$.

Corollary 6. *Under the situation of Theorem 4 assume moreover that* $\dim F_i<\dim M/2$ *for all i. Then*

$$\sum_{i=1}^{l}\sum_{s=0}^{[f_i/2]} j_!(v_s(F_i)^2)=0$$

and $\theta(\sigma)\in H^*_{Z_2}(M^2;Z_2)$ is characterized by the conditions

a) $\theta(\sigma)\in\pi_!$*-image*

and

b) $\pi^*\theta(\sigma)=\Delta_!(1)$.

Corollary 7. *Let M be an m-manifold which is a Z_2-homology sphere and σ an involution on M. Then, in the usual homology theory $H^*(\ ;Z_2)$, the element $\theta(\sigma)\in H^m_{Z_2}(M^2;Z_2)$ is given by*

$$\theta(\sigma)=\begin{cases}\pi_!(1\times\mu) & \text{if } \sigma \text{ is not trivial,}\\ w_1^m+\pi_!(1\times\mu) & \text{if } \sigma \text{ is trivial,}\end{cases}$$

where $\mu\in H^m(M;Z_2)$ is the cofundamental class.

Now let N be another manifold with an involution τ and $f:N\to M$ a continuous map. We put

$$A(f)=\{y\mid y\in N,\ f\tau(y)=\sigma f(y)\}$$

and define an equivariant map $\hat f:N\to M^2$ by $\hat f(y)=(f(y),f\tau(y))$. The following is fundamental for our theorems of the Borsuk-Ulam type.

Theorem 8. *If $A(f)=\phi$ then the class $\hat f^*\theta(\sigma)\in h^m_{Z_2}(N)$ vanishes.*

Corollary 9. *Let $\bar f$ denote the restriction of f on the fixed point set $F(\tau)$ of τ. Suppose that we have*

$$\bar f^*\Big(\sum_{i=1}^{l}\sum_{s=0}^{[f_i/2]} j_!(v_s(F_i)^2)\Big)\neq 0$$

in $H^*_{Z_2}(pt)\otimes H^*(F(\tau);Z_2)$ then the set $A(f)$ is not empty.

When the involution τ on N is free the module $h^*_{Z_2}(N)$ is canonically identified with $h^*(N/Z_2)$.

Corollary 10. *Let M and N be manifolds of the same dimension m. Let σ be an involution on M such that* $\dim F_i<\dfrac{m}{2}$ *for all components F_i of the fixed point set of σ. Let τ be a free involution on N and $f:N\to M$ a continuous map. Then, in the usual cohomology,* *the evaluation of the class* $\hat{f}^*\theta(\sigma) \in H^m(N/\boldsymbol{Z}_2)$ *on the fundamental class* $[N/\boldsymbol{Z}_2]$ *is given by*

$$\langle [N/\boldsymbol{Z}_2], \hat{f}^*\theta(\sigma)\rangle = \hat{\chi}(f)$$

where $\hat{\chi}(f)$ *is the equivariant Lefschetz number of* f *as defined in* [3]. *Consequently if* $\hat{\chi}(f)\neq 0$, *then* $A(f)\neq\phi$.

Corollary 11. *Let* M *be an* m*-manifold which is a* $\boldsymbol{Z}_2$*-homology sphere with an involution* σ. *Let* N *be an* m*-manifold with a free involution* τ *and* $f: N\to M$ *a map. Then we have*

$$\langle [N/\boldsymbol{Z}_2], \hat{f}^*\theta(\sigma)\rangle = \begin{cases} 1+\deg f & \text{if } \sigma \text{ is trivial,} \\ \deg f & \text{if } \sigma \text{ is not trivial.} \end{cases}$$

Consequently if σ *is not trivial and* $\deg f\neq 0$, *then* $A(f)\neq\phi$.

2. Indication of proofs. Lemma 1 is a consequence of the following structure theorem for $h^*_{\boldsymbol{Z}_2}(M^2)$ and a localization theorem due to tom Dieck [2] applied to the diagonal map d.

Theorem 12. *In* $h^*_{\boldsymbol{Z}_2}(M^2)$ *the union* $\bigcup_{k\geq 1}(\cup\ w_1^k$*-kernel) coincides with* $\pi_!$*-image which is isomorphic to* $h^*(M^2)/h^*(M^2)^T$ *through* $\pi_!$. *The homorphism* π^* *restricted on* $\pi_!$*-image is injective. The quotient* $h^*_{\boldsymbol{Z}_2}(M^2)/(\pi_!$*-image) is a free* $h^*_{\boldsymbol{Z}_2}(pt)$*-module and is generated by* P*-image. Its rank is equal to the rank of the* $h^*(pt)$*-module* $h^*(M)$.

Theorem 12 is proved using the Gysin exact sequence of the double covering $\pi: S^\infty\times M^2\to S^\infty\times_{\boldsymbol{Z}_2} M^2$ and the following properties of $\pi_!, \pi^*$ and P:

$$\pi^*\pi_!(u\times v)=u\times v+v\times u,$$
$$\pi^*P(u)=u\times u.$$

Part i) of Theorem 2 follows from the commutativity of the diagram

$$\begin{array}{ccc} h^*(M) & \xrightarrow{\Delta_!} & h^*(M^2) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi_*} \\ h^*_{\boldsymbol{Z}_2}(M) & \xrightarrow{\Delta_!} & h^*_{\boldsymbol{Z}_2}(M^2) \end{array}$$

which holds since π is a covering projection.

In order to prove Part ii) we consider the submanifolds $\Delta(M)$ and $d(M)$ of M^2. They are invariant under the action T. Their intersection is canonically identified with F. Let $j': F\subset\Delta(M)$ and $j: F\subset d(M)$ be the inclusions. Let $\nu_{j'}$ and ν_d be the normal bundles of j' and d respectively. We see that $\Delta(M)$ and $d(M)$ cut each other cleanly along F, that is, $\nu_{j'}$ is a subbundle of $j^*\nu_d$. Thus we have the excess bundle $E=j^*\nu_d/\nu_{j'}$ and it follows from the clean intersection formula (cf. [6]) that

$$d^*\Delta_!(1)=j_!(e(E))$$

where $e(E)\in h^*_{\boldsymbol{Z}_2}(pt)\otimes_{h^*(pt)} h^*(F)$ is the h^*-theory Euler class of the bundle E with $\boldsymbol{Z}_2$-action. In our situation we have

Lemma 13. *The bundle E is isomorphic to the normal bundle $\nu_{d'}$ of the diagonal map $d' : F \to F^2$ where the Z_2-action on $\nu_{d'}$ is induced from T.*

From Lemma 13 and the clean intersection formula applied to the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{d'} & F^2 \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j^2} \\ M & \xrightarrow{d} & M^2 \end{array}$$

we infer that

$$(*) \qquad d^*\Delta_!(1) = d^*\left(\frac{(j^2)_!(d'_!(1)^2)}{d_!(1)}\right),$$

in $(S^{-1}h^*_{Z_2}(pt) \underset{h^*(pt)}{\otimes} h^*(M)$. But we have a formula due to Nakaoka [5] which expresses $d_!(1)$ in terms of $v_\alpha(M), P_0$ and $a^\alpha(w_1)$ and a similar one for $d'_!(1)$. Using these in $(*)$ we obtain the formula in Part ii) of Theorem 2.

Finally Theorem 8 follows from the fact that $\hat{f}^*\theta(\sigma)$ is the Poincaré dual (in the equivariant cohomology) of $\hat{f}^{-1}(\Delta(M)) = A(f)$ in N.

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