121. *Kähler Metrics on Elliptic Surfaces*

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The purpose of this note is to outline a proof of the following

Theorem. *An elliptic surface admits a Kähler metric if and only if its first Betti number is even.*

Professor Kodaira raised a problem: *Does every compact analytic surface with an even first Betti number admit a Kähler metric?*

Our theorem solves this problem in the affirmative except the case in which the surface is a $K3$ surface.

1. Some cohomology groups on elliptic surfaces. Let $\Phi: B\to\Delta$ be an elliptic surface with a section $o: \Delta\to B$. We employ the notation of Kodaira [2]. Thus J, G and $\mathfrak{f}$ denote, respectively, the functional invariant of B, the homological invariant of B and the normal bundle of $o(\Delta)$ in B.

The following proposition is due to Shioda [5].

Proposition 1. *There exist canonical homomorphisms*

$$\alpha: H^1(\Delta, G)\to j^*(H^2(B, \boldsymbol{Z}))\subset H^2(B, \mathcal{O}),$$
$$\beta: H^1(\Delta, \mathcal{O}(\mathfrak{f}))\to H^2(B, \mathcal{O}),$$

such that

(i) $\operatorname{Im}\alpha$ *is a commensurable subgroup of* $j^*(H^2(B, \boldsymbol{Z}))$,

(ii) β *is an isomorphism,*

(iii) *the diagram*

$$\begin{array}{ccc} H^1(\Delta, G) & \xrightarrow{i^*} & H^1(\Delta, \mathcal{O}(\mathfrak{f})) \\ \alpha\downarrow & & \downarrow\beta \\ j^*(H^2(B, \boldsymbol{Z})) & \hookrightarrow & H^2(B, \mathcal{O}) \end{array}$$

is commutative, where i^* and j^* denote the natural homorphisms induced by the canonical injections $i: G\to\mathcal{O}(\mathfrak{f})$ and $j: \boldsymbol{Z}\to\mathcal{O}$, respectively.

Proof. We have canonical isomorphisms

$$G\xrightarrow{\approx}R^1\Phi_*(\boldsymbol{Z}),$$
$$\mathcal{O}(\mathfrak{f})\xrightarrow{\approx}R^1\Phi_*(\mathcal{O}_B),$$

and, moreover, i is compatible with j^* through the isomorphisms. We shall identify G, $\mathcal{O}(\mathfrak{f})$ and i, respectively, with $R^1\Phi_*(\boldsymbol{Z})$, $R^1\Phi_*(\mathcal{O}_B)$ and j^*. Let us consider the Leray spectral sequences:

$$'E_2^{pq}=H^p(\Delta, R^q\Phi_*(\boldsymbol{Z}))\Rightarrow H^{p+q}(B, \boldsymbol{Z}),$$
$$''E_2^{pq}=H^p(\Delta, R^q\Phi_*(\mathcal{O}_B))\Rightarrow H^{p+q}(B, \mathcal{O}_B).$$

Since $\Phi: B\to\Delta$ is a flat (2, 1)-fibre manifold, it is trivial that $'E_r$ degenerates for $r\geqq 3$, and that $''E_r$ degenerates for $r\geqq 2$. We thus obtain the canonical isomorphism

$$\beta: H^1(\Delta, R^1\Phi_*(\mathcal{O}_B)) \overset{\approx}{\to} H^2(B, \mathcal{O}_B)$$

and the canonical injection

$$H^2(\Delta, \boldsymbol{Z})/\mathrm{Im}\, 'd_2^{0,1} \overset{\iota}{\hookrightarrow} H^2(B, \boldsymbol{Z}).$$

In virtue of the functoriality of the Leray sequences, the diagram

$$\begin{array}{ccc} H^2(\Delta, \boldsymbol{Z})/\mathrm{Im}\, 'd_2^{0,1} & \overset{\iota}{\hookrightarrow} & H^2(B, \boldsymbol{Z}) \\ \downarrow & & \downarrow j^* \\ 0 = H^2(\Delta, \mathcal{O})/\mathrm{Im}\, ''d_2^{0,1} & \hookrightarrow & H^2(B, \mathcal{O}) \end{array}$$

is commutative, and *a fortiori* $j^*\circ\iota$ is a zero map. Now we define the natural homomorphism

$$\alpha: H^1(\Delta, R^1\Phi_*(\boldsymbol{Z})) \to j^*(H^2(B, \boldsymbol{Z})).$$

One sees that the condition (iii) is automatically satisfied. To prove (i), we consider the following two cases:

(a) *The case where the functional invariant J is not constant.* Let $\rho(B)$ denote the Picard number of B. Then

$$\mathrm{rank}\ \mathrm{Im}\, i^* = b_2(B) - \rho(B) = \mathrm{rank}\ j^*(H^2(B, \boldsymbol{Z})).$$

(See Ogg [4] and Shioda [5].) This proves the assertion.

(b) *The case where J is constant* (cf. Deligne [1]). $\Phi: B\to\Delta$ has a structure of an abelian scheme over Δ with the identity o. The multiplication by an integer m is an endomorphism μ_m over Δ. μ_m^* acts naturally on $H^p(\Delta, R^q\Phi_*(Q))$ as the multiplication by m^q. Since μ_m^* and $'d_2$ are commutative, the diagram

$$\begin{array}{ccc} H^p(\Delta, R^q\Phi_*(\boldsymbol{Q})) & \overset{'d_2}{\longrightarrow} & H^{p+2}(\Delta, R^{q-1}\Phi_*(\boldsymbol{Q})) \\ \times m^q \downarrow & & \downarrow \times m^{q-1} \\ H^p(\Delta, R^q\Phi_*(\boldsymbol{Q})) & \overset{'d_2}{\longrightarrow} & H^{p+2}(\Delta, R^{q-1}\Phi_*(\boldsymbol{Q})) \end{array}$$

is commutative. This implies that $'E_2$ degenerates. Thus we obtain the following isomorphism

$$\bigoplus_{p+q=r} H^p(\Delta, R^q\Phi_*(\boldsymbol{Q})) \overset{\approx}{\to} H^r(B, \boldsymbol{Q}).$$

On the other hand, μ_m^* acts on $H^2(B, \mathcal{O}) = H^1(\Delta, R^1\Phi_*(\mathcal{O}))$ as the multiplication by m. Hence $\alpha: H^1(\Delta, R^1\Phi_*(\boldsymbol{Q})) \to j^*(H^2(B, \boldsymbol{Q}))$ is surjective, which completes the proof. Q.E.D.

As a corollary we obtain the following

Proposition 2. $i^*(H^1(\Delta, G))\otimes\boldsymbol{R} = H^1(\Delta, \mathcal{O}(\mathfrak{f}))$.

Proof. In fact $j^*(H^2(B, \boldsymbol{Z}))\otimes\boldsymbol{R} = H^2(B, \mathcal{O})$, because B is a compact Kähler manifold. Q.E.D.

2. Proof of the theorem. Let $\Phi: B\to\Delta$ be an elliptic surface with a section $o: \Delta\to B$. $\Phi^\# = \Phi | B^\#: B^\#\to\Delta$ has the structure of a (non-proper) abelian scheme over Δ with the identity o, and the multiplication by an integer m is an endomorphism μ_m of $B^{\sharp}$ over Δ. Evidently μ_m can be considered as a rational endomorphism of B over Δ with a finite set of fundamental points I.

Proposition 3. *$\mu_m|(B-I): B-I\to B$ is everywhere of maximal rank.*

Proof. Let R be the ramification locus of μ_m. Then $K_{B-I}=\mu_m^* K_B + [R]$. Observing the following commutative diagram

$$\begin{array}{ccccc} & & B-I & & \\ & \swarrow^{\iota} & \downarrow{\scriptstyle \Phi'} & \searrow^{\mu_m} & \\ B & & & & B \\ & \searrow_{\Phi} & \Delta & \swarrow_{\Phi} & \end{array}$$

we have

$$K_{B-I}=\iota^* K_B=\iota^*\Phi^*(K_\Delta-\mathfrak{f})=\Phi'^*(K_\Delta-\mathfrak{f})=\mu_m^*\Phi^*(K_\Delta-\mathfrak{f})=\mu_m^* K_B.$$

Hence $[R]$ is trivial. This proves $R=0$. Q.E.D.

For $\gamma\in H^1(\Delta, \Omega(B_0^{\sharp}))$, B^{γ} is defined to be an elliptic surface over Δ and $\mu_m: B^{\gamma}\to B^{m\gamma}$ is a meromorphic mapping with a finite set of fundamental points I^{γ}. We prove similarly that μ_m: $B^{\gamma}-I^{\gamma}\to B^{m\gamma}$ is everywhere of maximal rank. Combined with the corollary to Proposition 1 in [3], this fact implies the following

Proposition 4. *B^{γ} is a Kähler surface if $B^{m\gamma}$ is a Kähler surface for some integer m.*

Next, we prove

Proposition 5. *Let $E\to\Delta$ be an elliptic surface free from multiple fibres. If the first Betti number is even, then E is a Kähler surface.*

Proof. We can express $E=B^{\gamma}$ for a suitable $\gamma\in H^1(\Delta, \Omega(B_0^{\sharp}))$ (cf. Kodaira [2], §§8 and 9). $b_1(E)$ is even if and only if the "*Chern class*" $c(\gamma)\in H^2(B, G)$ is an element of finite order g in $H^2(B, G)$. It suffices therefore to prove that $B^{m\gamma}$ is a Kähler surface for some integer m. Replacing γ by $g\gamma$, we may assume that $c(\gamma)=0$ and that $\gamma\in H^1(\Delta, \mathcal{O}(\mathfrak{f}))/H^1(\Delta, G)$. From Proposition 2 we infer that $\{k\gamma\}_{k=1,2,\dots}$ has a subsequence $\{\gamma_k\}_{k=1,2,\dots}$ which converges to γ_0 with a finite order. Hence for a suitable $m\in \boldsymbol{Z}$, $B^{m\gamma}$ is a small deformation B^{γ_0}. This proves the assertion. Q.E.D.

Let $E\xrightarrow{\Phi}\Delta$ be an elliptic surface. Then we can find a finite Galois covering $\tilde{\Delta}\to\Delta$ such that

(i) the induced fibre variety $\tilde{E}\xrightarrow{\tilde{\Phi}}\tilde{\Delta}$ is a non-singular elliptic surface free from multiple fibres,

(ii) the induced mapping $p:\tilde{E}\to E$ is a finite Galois covering whose branch locus is a regular fibre E_0.

(For a proof, see Kodaira [2], § 6.) Therefore, by the aid of Proposition 2 in [3], the theorem is an immediate corollary to the following

Lemma. *If $b_1(E)$ is even, then $b_1(\tilde{E})$ is also even.*

Proof. Let $\Gamma = \mathrm{Aut}\,(\tilde{\Delta}/\Delta)$ denote the Galois group. By considering the Leray spectral sequences, we have the commutative diagram:

$$\begin{array}{ccccc} H^0(\tilde{\Delta}, R^1\tilde{\Phi}_*(Q))^\Gamma \cong & H^0(\Delta, R^1\Phi_*(Q)) & \xrightarrow{q = {'d}_2^{0,1}} & H^2(\Delta, Q) \\ \searrow & \downarrow \pi^* & & \downarrow \\ & H^0(\tilde{\Delta}, R^1\tilde{\Phi}_*(Q)) & \xrightarrow{\tilde{q} = {'\tilde{d}}_2^{0,1}} & H^2(\tilde{\Delta}, Q). \end{array}$$

Now assume that $b_1(\tilde{E})$ is odd. Then we infer that $R^1\tilde{\Phi}_*(Q)$ is trivial (cf. [2], § 9) and that $\tilde{q}$ is surjective. For any $\xi \in H^2(\tilde{\Delta}, Q)$ there exists $\psi \in H^0(\tilde{\Delta}, R^1\tilde{\Phi}_*(Q))$ such that $\xi = \tilde{q}(\psi)$. Because Γ acts trivially on $H^2(\tilde{\Delta}, Q)$, we have

$$\xi = \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \gamma^*\xi = \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} q(\gamma^*\psi).$$

This proves that q is surjective. Moreover it turns out that $R^1\Phi_*Q$ is trivial. In fact, since $R^1\tilde{\Phi}_*Q|_P$ ($P \in \tilde{\Delta}$) contains a 1-dimensional Γ-invariant subspace, $\gamma \in \Gamma$ has the following matrix expression:

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in SL(2, Z).$$

On the other hand Γ is a finite group. Hence γ acts on $R^1\tilde{\Phi}_*Q$ trivially. Thus we have proved that dim Ker $q = 1$ and therefore $b_1(E) = b_1(\Delta) + 1$ $=$ odd. Q.E.D.

References

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