190. *Characters of Finite Groups with Split* (B, N)*-Pairs*

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§ 1. In our previous paper [4], we discussed the irreducibility of characters of the finite general unitary group $GU(n, q^2)$ induced by those of a direct product of the finite general linear group $GL(k, q^2)$ and $GU(n-2k, q^2)$. Recently we were suggested by Professor C. W. Curtis that one would be able to get a similar result for finite groups with split (B, N)-pairs. Using the results of intersections of parabolic subgroups in a paper by Curtis [2], we could generalize the result in our paper [4]. Note that this is a special case of Theorem 3.5 due to Curtis [2].

I wish to thank Professor Curtis for his suggestion to me on this problem and also for the generous use of his preprint [2].

By a character of a group, we mean a rational integral combination of its complex irreducible characters. Standard notations for finite group theory and character theory will be used.

Let G be a finite group with a split (B, N)-pair of characteristic p, for some prime p, and Coxeter system (W, R). Let P_J be a standard maximal parabolic subgroup of G, L_J the standard Levi factor of P_J for some $J \subseteq R$. Then P_J has a semi-direct decomposition $P_J = L_J V_J$ of $V_J = O_p(P_J)$ by L_J, which we call the Levi decomposition of P_J. If χ is an irreducible character of L_J, then we can extend χ to an irreducible character $\tilde{\chi}$ of P_J, by putting $\tilde{\chi}(lv) = \chi(l)$ for $l \in L_J, v \in V_J$. We shall now prove the following

Theorem. *Let $W_{J,J}$ be the set of distinguished (W_J, W_J)-double coset representatives of W. Assume that* (i) *χ is not a self-conjugate and* (ii) *no kernel of irreducible constituents of the restriction of χ to $L_J \cap {}^wP_J$ contains $L_J \cap {}^wV_J$ whenever $L_J \neq {}^wL_J$ for $w \in W_{J,J}$. Then the character $\tilde{\chi}^G$ of G induced by $\tilde{\chi}$ is irreducible.*

In order to prove this theorem, we must calculate the scalar product $(\tilde{\chi}^G, \tilde{\chi}^G)_G$. To do this, it will be necessary to derive some informations of parabolic subgroups. In § 2, we shall state several results about intersections of parabolic subgroups due to Curtis [2]. The theorem is proved in § 3. The proof is a simple combination of lemmas in § 2, § 3.

§ 2. Let (G, B, N, W, R) be as in § 1. Then W is isomorphic to the Weyl group $W(\varDelta)$ of a uniquely determined root system $\varDelta$, such that the set R corresponds to a set of fundamental reflections of $W(\Delta)$ with respect to a set of simple roots $\Pi=\{\alpha_1, \cdots, \alpha_n\}$ in Δ. We identify W with $W(\Delta)$ and R with the set of fundamental reflections $\{r_1, \cdots, r_n\}$. We denote by $l(w)$ the length of w as an element of (W, R). The set of positive (resp. negative) roots in Δ with respect to Π is denoted by Δ_+ (resp. Δ_-). We also put, for $w \in W$, $\Delta_w^+=\Delta_+ \cap w^{-1}(\Delta_+)$, $\Delta_w^-=\Delta_+ \cap w^{-1}(\Delta_-)$. Moreover let w_R denote the unique element of W such that $w_R(\Delta_+)=\Delta_-$. Then w_R is an involution.

Now put $T=B\cap N$. As is well-known, $T \trianglelefteq N$, $N/T=W$ and B is a semi-direct product UT of $U=O_p(B)$ by T. Let $\{n_w\}$ be a fixed set of coset representatives of T in N, such that n_wT corresponds to $w \in W$. We may write BwB for Bn_wB and write H^w (resp. ${}^wH$) instead of H^{n_w} (resp. ${}^{n_w}H$) for a subgroup H of G normalized by T. We also put $U_{\alpha_i}=U\cap U^{w_R r_i}$. Note that T normalizes the $\{U_{\alpha_i}; \alpha_i \in \Pi\}$, so that W acts on the set of N-conjugates of the $\{U_{\alpha_i}; \alpha_i \in \Pi\}$. Thus we can speak unambiguously of root subgroups U_α for $\alpha \in \Delta$ and have the familiar rule ${}^wU_\alpha=U_{w(\alpha)}$ for $w \in W, \alpha \in \Delta$. Then U is generated by U_α corresponding to $\alpha \in \Delta_+$.

For $J \subseteq R$, we denote by W_J the parabolic subgroup of W generated by J, and by P_J the corresponding standard parabolic subgroup of G, given by $P_J=BW_JB$. Let Π_J be the set of simple roots corresponding to J, Δ_J the root system generated by Π_J and put $\Delta_{J,+}=\Delta_+\cap\Delta_J$, $\Delta_{J,-}=\Delta_-\cap\Delta_J$. Let w_J denote the unique element of W_J such that $w_J(\Delta_{J,+})=\Delta_{J,-}$. Then w_J is an involution and (W_J, J) is a Coxeter system.

Next two lemmas are elementary.

Lemma 2.1. *Let $w \in W$. Then*

(a) $l(r_iw)=l(w)\pm 1$ *if* $w^{-1}(\alpha_i) \in \Delta_\pm$,

(b) $l(wr_i)=l(w)\pm 1$ *if* $w(\alpha_i) \in \Delta_\pm$,

(c) $Br_iBwB \subseteq Br_iwB$ *if* $l(r_iw) \geq l(w)$,

(d) $Br_iBwB \cap BwB \neq \emptyset$ *if* $l(r_iw) \leq l(w)$.

Proof. See [1].

Lemma 2.2. *Let $J \subseteq R$ and $w \in W_J$. Then $w(\Delta_+-\Delta_{J,+}) \subseteq \Delta_+$. In particular, $\Delta_{w_J}^+=\Delta_+-\Delta_{J,+}$ and $\Delta_{w_J}^-=\Delta_{J,+}$.*

Proof. As $r_i(\Delta_+-\{\alpha_i\})=\Delta_+-\{\alpha_i\}$, we have $w(\Delta_+-\Delta_{J,+}) \subseteq \Delta_+$. Hence $\Delta_w^- \subseteq \Delta_{J,+}$. However the definition of w_J implies that $\Delta_{w_J}^+ \subseteq \Delta_+-\Delta_{J,+}$ and $\Delta_{J,+} \subseteq \Delta_{w_J}^-$. This completes the proof.

Let L_J be the subgroup of P_J generated by T and U_α corresponding to $\alpha \in \Delta_J$, which is called the standard Levi factor of P_J and $P_J=L_JV_J$ be the Levi decomposition of P_J. Thus V_J is the unique maximal normal p-subgroup of P_J generated by U_α corresponding to $\alpha \in \Delta_+-\Delta_{J,+}$, $P_J=N_G(V_J)$ and (L_J, B_J, N_J, W_J, J) is a finite group with a split (B, N)-pair, where $B_J=B\cap L_J$, $N_J=N\cap L_J$. Moreover we have $B=B_JV_J$.

For $J, J' \subseteq R$, let $W_{J,J'}$ be the set of distinguished $(W_J, W_{J'})$-double coset representatives of W, that is, $w \in W_{J,J'}$ satisfies $w(\alpha) \in \Delta_+$, $w^{-1}(\beta) \in \Delta_+$ for $\alpha \in \Pi_{J'}$, $\beta \in \Pi_J$ and w is the element of W of the shortest length in $W_J w W_{J'}$. We now put $K = J \cap {}^wJ'$ for a fixed element w of $W_{J,J'}$. Note that $G = \bigcup_{w \in W_{J,J'}} P_J w P_{J'}$ (disjoint union) and $W_J \cap {}^wW_{J'} = W_K$. For the rest of this section, these notations will be used.

The following lemma is of importance in the later development.

Lemma 2.3. (a) $\Pi_J \cap w(\Pi_{J'}) = \Pi_K$, $\Delta_J \cap w(\Delta_{J'}) = \Delta_K$,

(b) $\Delta_{J,+} \subseteq w(\Delta_+)$, $w(\Delta_{J',+}) \subseteq \Delta_+$,

(c) $\Delta_{K,+} = \Delta_{J,+} \cap w(\Delta_{J',+})$,

(d) $\Delta^+_{w_K} - \Delta^+_{w_J} \subseteq w(\Delta^+_{w_{J'}}) \cap \Delta_J$.

Proof. (a) $\alpha = w(\beta)$ for $\alpha \in \Pi_J$, $\beta \in \Pi_{J'}$ if and only if $w_\alpha = {}^w w_\beta \in W_J \cap {}^wW_{J'} = W_K$. So (a) is clear. (b) As $l(rw) > l(w)$ and $l(wr') > l(w)$ for $r \in J$, $r' \in J'$, we have $w^{-1}(\Delta_{J,+}) \subseteq \Delta_+$ and $w(\Delta_{J',+}) \subseteq \Delta_+$ by Lemma 2.1. (a) and (b) implies that $w(\Delta_+ - \Delta_{J',+}) \cap \Delta_K = \emptyset$. Hence we get (c). (d) If $\alpha \in \Delta^+_{w_K} - \Delta^+_{w_J}$, then $\alpha \in \Delta_{J,+} \cap w(\Delta_+)$ and $\alpha \notin \Delta_{K,+}$ by Lemma 2.2. Therefore (c) implies (d) and so the lemma is proved.

We can now derive some consequences for intersections of parabolic subgroups of G, which are based on preceding lemmas.

Lemma 2.4. $P_K = (P_J \cap {}^wP_{J'})V_J$.

Proof. By Lemma 2.3 (b) we have $B_J \leq L_J \cap {}^wB \leq P_J \cap {}^wP_{J'}$ and so $B \leq (P_J \cap {}^wP_{J'})V_J$. Hence $(P_J \cap {}^wP_{J'})V_J = P_I$ for some $I \subseteq R$. As $l(rw) > l(w)$ for $r \in J$, we have $rBw \subseteq BrwB$ by Lemma 2.1 (c). Then, for $w_1 \in W_J$, it is easy to see that $Bw_1BwB \subseteq Bw_1wB$, because $l(w_1w) = l(w_1) + l(w)$, etc. By a similar reason, $BwBw_2B \subseteq Bww_2B$ for $w_2 \in W_{J'}$. Hence $aw_1bw = wcw_2d \in Bw_1wB \cap Bww_2B$, where $a, b, c, d \in B$, $w_1 \in W_J$, $w_2 \in W_{J'}$. Thus $Bw_1wB \cap Bww_2B \neq \emptyset$. Then $w_1w = ww_2$ and so $(P_J \cap {}^wP_{J'})V_J \leq B(W_J \cap {}^wW_{J'})B = P_K$. The reverse inclusion is clear.

Lemma 2.5. (a) $V_K = (L_J \cap {}^wV_{J'})V_J$,

(b) $P_J \cap {}^wV_{J'} = (L_J \cap {}^wV_{J'})(V_J \cap {}^wV_{J'})$,

(c) $V_J \cap {}^wP_{J'} = (V_J \cap {}^wL_{J'})(V_J \cap {}^wV_{J'})$,

(d) *$L_J \cap {}^wP_{J'}$ is a standard parabolic subgroup of L_J; in fact, $L_J \cap {}^wP_{J'} = P_K \cap L_J$ and $L_J \cap {}^wP_{J'} = L_K(L_J \cap {}^wV_{J'})$ is a Levi decomposition of $L_J \cap {}^wP_{J'}$ with $L_J \cap {}^wV_{J'} = O_p(L_J \cap {}^wP_{J'})$.*

Proof. (a) As V_J is normalized by $L_J \cap {}^wV_{J'}$, $(L_J \cap {}^wV_{J'})V_J$ is a group. ${}^wV_{J'}$ is the group generated by ${}^wU_\alpha$ corresponding to $\alpha \in \Delta^+_{w_{J'}}$ and so $V_K \leq (L_J \cap {}^wV_{J'})V_J$ by Lemma 2.3 (d). Suppose $\alpha \in w(\Delta^+_{w_{J'}}) \cap \Delta_J$. Then we have $\alpha \in \Delta_+$, $\alpha \notin \Delta_K$ by Lemma 2.3 (a)(b). Hence we have $\alpha \in \Delta^+_{w_K}$ by Lemma 2.2. Thus $L_J \cap {}^wV_{J'} \leq V_K$. Clearly $V_J \leq V_K$ by Lemma 2.3 (a). Hence we get (a). (b) As $(P_J \cap {}^wV_{J'})V_J \leq U$, $(P_J \cap {}^wV_{J'})V_J$ is a normal p-subgroup of P_K and so $(P_J \cap {}^wV_{J'})V_J \leq O_p(P_K) = V_K$. Each element $x \in P_J \cap {}^wV_{J'}$ is uniquely expressible in the form $x = yz$ with

$y \in L_J, z \in V_J$. As $x \in V_K$, we have $y \in {}^wV_{J'}$ by Lemma 2.3 (d). Hence $z = y^{-1}x \in {}^wV_{J'}$. Thus $P_J \cap {}^wV_{J'} \leq (L_J \cap {}^wV_{J'})(V_J \cap {}^wV_{J'})$. The reverse inclusion is clear. (c) As $w^{-1} \in W_{J',J}$, (b) implies (c). (d) It is easy to see that $P_K \cap L_J$ is a standard parabolic subgroup of L_J with Levi factor L_K and $V_K \cap L_J = O_p(P_K \cap L_J)$. We also have $V_K \cap L_J = L_J \cap {}^wV_{J'}$ by (a) and $L_K \leq L_J \cap {}^wL_{J'}$ by Lemma 2.3 (a). Hence $P_K \cap L_J \leq L_J \cap {}^wP_{J'}$. On the other hand, $L_J \cap {}^wP_{J'} \leq P_K$ by Lemma 2.4. Therefore $L_J \cap {}^wP_{J'} = P_K \cap L_J$. This completes the proof.

Lemma 2.6. *The following conditions are equivalent.*

(a) $L_J \cap {}^wV_{J'} = 1$.

(b) $L_J \leq {}^wL_{J'}$.

Proof. If (a) holds, then $V_K = V_J$ by Lemma 2.5 (a). Hence $P_K = P_J$ and so $W_K = W_J$. Thus $\Delta_K = \Delta_J$. This implies (b) by Lemma 2.3 (a). If (b) holds, then $L_J \cap {}^wV_{J'} \leq {}^wL_{J'} \cap {}^wV_{J'} = 1$ and the result follows.

Lemma 2.7. $P_J \cap {}^wP_{J'} = L_K(L_J \cap {}^wV_{J'})(V_J \cap {}^wL_{J'})(V_J \cap {}^wV_{J'})$. *In particular,* $P_J \cap {}^wP_{J'} = L_J(V_J \cap {}^wV_{J'})$ *if* $L_J \leq {}^wL_{J'}$.

Proof. By Lemmas 2.4, 2.5 (a) we have $P_J \cap {}^wP_{J'} \leq L_K(L_J \cap {}^wV_{J'})V_J$ and so $P_J \cap {}^wP_{J'} = L_K(L_J \cap {}^wV_{J'})(V_J \cap {}^wP_{J'})$. Hence the first part is proved by Lemma 2.5 (c). Suppose $L_J \leq {}^wL_{J'}$. By Lemma 2.6 we have $P_J \cap {}^wP_{J'} = L_K(V_J \cap {}^wV_{J'})$. But it follows from the proof of Lemma 2.6 that $\Delta_K = \Delta_J$. Therefore $L_K = L_J$. This completes the proof.

§ 3. We first begin with next two lemmas which are of importance for the applications of character theory.

Lemma 3.1. *Let H be a subgroup of a group G, χ an irreducible character of H. Let $\{g_i\}$ be the set of (H,H)-double coset representatives of G and put $H_i = H \cap {}^{g_i}H$. Then*

$$(\chi^G, \chi^G)_G = \sum_i (\chi, {}^{g_i}\chi)_{H_i}.$$

Proof. This is a special case of the well-known result, due to Mackey (see [3]).

Lemma 3.2. *Let H be a normal subgroup of a group G, χ an irreducible character of G. Assume that the kernel of χ does not contain H. Then, for $g \in G$, $\sum_{h \in H} \chi(gh) = 0$.*

Proof. It follows from the assumption and Frobenious reciprocity theorem that $(\chi_H, 1_H)_H = (\chi, 1_H^G)_G = 0$, where 1_H is the principal character of H. We now denote by χ the matrix representation of G which affords χ and put $S = \sum_{h \in H} \chi(h)$. Since $H \trianglelefteq G$, $S\chi(g) = \chi(g)S$ for $g \in G$. Hence Schur's lemma asserts that S is a scalar matrix and so $S = 0$. Therefore taking the trace, we have $\sum_{h \in H} \chi(gh) = 0$, as required.

Throughout the rest of this section, we assume the notations of our theorem. For shortness, write P, L, V instead of P_J, L_J, V_J respectively. For a fixed element $w \in W_{J,J}$, we denote by I_w the scalar product $(\tilde{\chi}, {}^w\tilde{\chi})_{P \cap {}^wP}$ and put $K = J \cap {}^wJ$.

3.3. *If* $L \neq {}^wL$, *then* $I_w = 0$.

Proof. By the canonical form for elements of $P \cap {}^wP$ established in Lemma 2.7, each element of $P \cap {}^wP$ has a unique expression in the form $xyzv$, where $x \in L_K, y \in L \cap {}^wV, z \in V \cap {}^wL, v \in V \cap {}^wV$. Hence we have

$$I_w = |V \cap {}^wV| \cdot |P \cap {}^wP|^{-1} \sum_{x,y,z} \tilde{\chi}(xyz)\overline{{}^w\tilde{\chi}(xyz)},$$

because $V \cap {}^wV$ is contained in the kernels of both characters $\tilde{\chi}, {}^w\tilde{\chi}$. Since $V \cap {}^wL, L \cap {}^wV$ are also contained in the kernels of $\tilde{\chi}, {}^w\tilde{\chi}$ respectively and L normalizes V, we have

$$I_w = |V \cap {}^wV| \cdot |P \cap {}^wP|^{-1} \sum_{x,y,z} \chi(xy)\overline{{}^w\chi(xz)},$$

where the sum is taken over all $x \in L_K, y \in L \cap {}^wV, z \in V \cap {}^wL$. As $L \cap {}^wV \trianglelefteq L_K(L \cap {}^wV)$, we have, by Lemma 3.2 and assumption (ii),

$$\sum_{y \in L \cap {}^wV} \chi(xy) = 0.$$

This implies $I_w = 0$.

3.4. *If* $L = {}^wL$ *and* $w \neq 1$, *then* $I_w = 0$.

Proof. By Lemma 2.7 it is easy to see that $I_w = (\chi, {}^w\chi)_L$. Hence it follows from assumption (i) that $I_w = 0$, as required.

3.5. **Conclusion.** By Lemmas 3.1, 3.3 and 3.4 $(\tilde{\chi}^G, \tilde{\chi}^G)_G = \sum_{w \in W_{J,J}} I_w = I_1 = 1$. Hence $\tilde{\chi}^G$ is irreducible. This completes the proof of our theorem.

§ 4. Let $GU_n = GU(n, q^2)$ be the group of all non-singular $n \times n$ matrices g with elements in the Galois field $GF(q^2)$ satisfying $g^* j_n g = j_n$, where g^* is the conjugate transpose of g and j_n is the matrix $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$ of degree n, $GL_n = GL(n, q^2)$ the group of all non-singular $n \times n$ matrices g with elements in $GF(q^2)$. We denote by $P_{n,k}$ the maximal parabolic subgroup of GU_n which consists of all matrices of the forms $\begin{bmatrix} a & d & e \\ & b & f \\ & & c \end{bmatrix}$ with $a \in GL_k, b \in GU_{n-2k}, a^* j_k c = j_k$.

Let take G, P_J in our theorem to be $GU_n, P_{n,k}$ respectively. Hence $L_J \cong GL_k \times GU_{n-2k}$. Then we can get, by our theorem, some families of irreducible characters of GU_n from those of GL_k and GU_{n-2k}.

Finally we give, for $n = 4, 5$, a list of the degrees of irreducible characters obtained by such a way.

Case of $n=4$: $q(q^2+1)(q^3+1)$, $(q^2+1)(q^3+1)$, $(q-1)(q^2+1)(q^3+1)$, $(q+1)(q^2+1)(q^3+1)$, $q^2(q+1)(q^3+1)$, $(q+1)(q^3+1)$, $(q+1)(q^2-1)(q^3+1)$.

Case of $n=5$: q^3s, $q(q-1)s$, s, $q(q^2-q+1)s$, $(q-1)(q^2-q+1)s$, $(q^2-q+1)s$, $(q^3+1)s$, $(q+1)(q^2-1)s$, q^2t, $(q^2-1)t$, t, where $s = (q^2+1)(q^5+1)$ and $t = (q^3+1)(q^5+1)$ (see [4]).

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