186. *On the Broadwell's Model for a Simple Discrete Velocity Gas*

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In this paper we discuss the question of the global existence of non-negative solutions satisfying the semilinear hyperbolic system of equations

$$u_t + u_x = \varepsilon(w^2 - uv) \qquad (1\text{-}1)$$

$$v_t - v_x = \varepsilon(w^2 - uv), \qquad (t, x) \in (0, +\infty) \times R \qquad (1\text{-}2)$$

$$w_t = -2\varepsilon(w^2 - uv) \qquad (1\text{-}3)$$

with the non-negative initial data

$$\begin{aligned} u(0, x) &= u_0(x) \\ v(0, x) &= v_0(x), \qquad x \in R \\ w(0, x) &= w_0(x). \end{aligned} \qquad (2)$$

This system was proposed by J. E. Broadwell as one of the simplest models of a dilute gas whose molecules move in the discrete state. In this model, u and v are the numbers of molecules per unit volume with the velocities $(1, 0, 0)$ and $(-1, 0, 0)$ respectively, w is that with the velocity $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$ and the gas motion is considered as one dimensional in x and homogeneous in y and z. A set (u, v, w) interacts only through binary collision with other molecules. As the collision coefficient ε is found to be proportional to the mutual-collision cross section, it may be taken as sufficiently small. A more detailed physical description of this model can be found in [1] and [3]. We remark that this approach gives the approximate solution of the Boltzmann equation in the meaning of restricting the molecular velocities to a finite set.

The local existence and uniqueness of the smooth or C^1-solution for the Cauchy problem (1) and (2) can be obtained as a classical result (see [2]). From now on, we denote the problem (1) and (2) by (C. Pr.).

As for the system (1), there exist the following relations which play an essential role to obtain the global solution of (C. Pr.); the conservation of mass:

$$(u + v + w)_t + (u - v)_x = 0 \qquad (3)$$

the conservation of momentum:

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$$(u-v)_t+(u+v)_x=0 \tag{4}$$

H-theorem:

$$\begin{aligned}&(u\log u+v\log v+w\log w)_t+(u\log u-v\log v)_x\\&\quad=-\varepsilon(w^2-uv)\log(w^2/uv)\leq 0.\end{aligned} \tag{5}$$

We examine some qualitative properties of the solution of (C. Pr.) in preparation for the proof of the global existence theorem. Throughout this paper, we suppose the existence of a C^1-local solution of (C. Pr.).

Lemma 1. *If the initial datum* $(u_0(x), v_0(x), w_0(x))$ *is non-negative (resp. positive), then the solution of* (C. Pr.) $(u(t,x), v(t,x), w(t,x))$ *is also non-negative (resp. positive).*

Proof. It is easy to prove this lemma, so we omit it.

Lemma 2. *Define* $f(z,c)$ *as*

$$f(z,c)=z\cdot\log(z/c)-z+c \tag{6}$$

for any positive constant c. *If* (C. Pr.) *has a positive solution* $(u(t,x), v(t,x), w(t,x))$, *then it follows that*

$$\begin{aligned}&\{f(u(t,x),u^0)+f(v(t,x),v^0)+f(w(t,x),w^0)\}_t\\&\quad+\{f(u(t,x),u^0)-f(v(t,x),v^0)\}_x\\&\quad=-\varepsilon(w^2-uv)\log(w^2/uv)(t,x),\end{aligned} \tag{7}$$

where (u^0, v^0, w^0) *is a positive equilibrium state of the system* (1), *i.e.,* $(w^0)^2=u^0v^0$.

Proof. The proof can be given as a direct consequence of (3) and (5).

Lemma 3. *Suppose that*

$$0\leq u_0(x), v_0(x), w_0(x)\leq K_0<+\infty \tag{8}$$

and

$$\int_{-\infty}^{\infty}\{u_0(x)+v_0(x)+w_0(x)\}dx=L_0<+\infty, \tag{9}$$

where K_0 *and* L_0 *are both positive constants. If* $\varepsilon<1/L_0$, *then the solution of* (C. Pr.) *has the following a priori estimate*;

$$0\leq u(t,x), v(t,x), w(t,x)\leq\frac{2K_0}{1-\varepsilon L_0}. \tag{10}$$

Proof. We first note that the following property is proved;

If $0\leq u(t,x), v(t,x)\leq K$ for any constant K with $w_0(x)\leq K$, then $0\leq w(t,x)\leq K$ by (1–3). On integrating (1–1) for C^1-solutions along the characteristic line $x-t=$const., and (1–2) along $x+t=$const. respectively, we have

$$u(t,x)=u_0(x-t)+\varepsilon\int_0^t(w^2-uv)(\tau, x-(t-\tau))d\tau \tag{11–1}$$

and

$$v(t,x)=v_0(x+t)+\varepsilon\int_0^t(w^2-uv)(\tau, x+(t-\tau))d\tau. \tag{11–2}$$

Addition of (11–1) to (11–2) gives

$$u(t,x)+v(t,x)$$
$$=u_0(x-t)+v_0(x+t)+\varepsilon\int_{x-t}^{x+t}(w^2-uv)(t-|x-\xi|,\xi)d\xi. \tag{12}$$

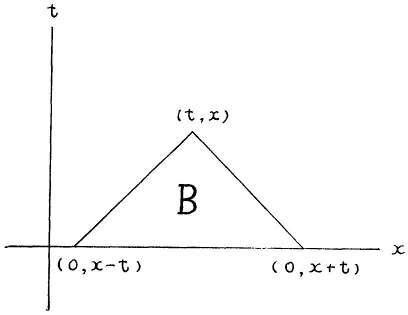
Now supposing that

$$0\leq u(t,x),\ v(t,x)\leq K, \tag{13}$$

then, by use of the non-negativity of $(u(t,x), v(t,x))$ and the inequality (8), (12) is, from the above property, estimated by

$$0\leq u(t,x)+v(t,x)\leq 2K_0+\varepsilon K\int_{x-t}^{x+t} w(t-|x-\xi|,\xi)d\xi. \tag{14}$$

Here we estimate the second term of the right hand side of (14). We integrate (3) over the triangle B.



Then we find that (3) gives

$$\int_{x-t}^{x+t} w(t-|x-\xi|,\xi)d\xi+2\int_{x}^{x+t}u(t+x-\xi,\xi)d\xi$$
$$+2\int_{x-t}^{x}v(t-x+\xi,\xi)d\xi \tag{15}$$
$$=\int_{x-t}^{x+t}(u_0+v_0+w_0)(\xi)d\xi.$$

Thus, as an immediate result of (15), we obtain the estimate of $w(t,x)$ such as

$$\int_{x-t}^{x+t} w(t-|x-\xi|,\xi)d\xi\leq L_0. \tag{16}$$

Substituting (16) into (14), we get

$$0\leq u(t,x)+v(t,x)\leq 2K_0+\varepsilon KL_0. \tag{17}$$

In this case, in order that (17) is consistent with (13), K must satisfy

$$2K_0+\varepsilon KL_0\leq K.$$

Hence, it is sufficient that ε and K satisfy

$$\varepsilon L_0<1 \quad\text{and}\quad K=\frac{2K_0}{1-\varepsilon L_0}. \tag{18}$$

Thus the proof is completed.

Lemma 4. *Suppose that*

$$0<\delta\leq u_0(x),\ v_0(x),\ w_0(x)\leq K_0<+\infty \tag{19}$$

and

$$\int_{-\infty}^{\infty}\{f(u_0(x),u^0)+f(v_0(x),v^0)+f(w_0(x),w^0)\}dx=E_0<+\infty, \tag{20}$$

where δ *and* E_0 *are both positive constants and* $\delta \leq u^0, v^0, w^0 < K_0$. *If* $\varepsilon < 1/2E_0$, *then the solution of* (C. Pr.) *has the following a priori estimate*;

$$0 < u(t,x), v(t,x), w(t,x) \leq \left(\frac{4}{1-2\varepsilon E_0}\right)^{(2(e-2)w^0/E_0)t} \cdot K_0.$$

Proof. The procedure of the proof is almost similar to that of Lemma 3. We first note

$$0 < u(t,x) + v(t,x) \leq 2K_0 + \varepsilon \int_{x-t}^{x+t} w^2(t-|x-\xi|, \xi)d\xi.$$

Supposing that

$$0 < u(t,x), v(t,x) \leq K_1 \qquad \text{for } t \in [0, t_1], \tag{21}$$

then it follows that

$$0 < u(t,x) + v(t,x)$$
$$\leq 2K_0 + 2\varepsilon t(w^0)^2 + \varepsilon(K_1 + w^0) \int_{x-t}^{x+t} |w(t-|x-\xi|, \xi) - w^0| d\xi, \qquad t \in [0, t_1].$$

Noting that

$$|w(t,x) - w^0| \leq f(w(t,x), w^0) + w^0(e-2)$$

and

$$\int_{x-t}^{x+t} f(w(t-|x-\xi|, \xi), w^0)d\xi \leq E_0,$$

then we can see that

$$0 < u(t,x) + v(t,x) \leq 2K_0 + 2\varepsilon t(w^0)^2 + \varepsilon(K_1 + w^0)\{E_0 + 2w^0(e-2)t\}. \tag{22}$$

In order that (22) is consistent with (21), the following inequality must hold;

$$2K_0 + 2\varepsilon t(w^0)^2 + \varepsilon(K_1 + w^0)\{E_0 + 2w^0(e-2)t\} \leq K_1. \tag{23}$$

Hence, if ε, t_1 and K_1 satisfy

$$t_1 = \frac{E^0}{2(e-2)w^0}, \qquad 2\varepsilon E_0 < 1$$

and

$$K_1 = \frac{2K_0 + \dfrac{\varepsilon w^0 E_0}{e-2} + 2\varepsilon w^0 E_0}{1 - 2\varepsilon E_0},$$

then we can see

$$0 < u(t,x), v(t,x), w(t,x) \leq K_1, \qquad (t,x) \in [0, t_1] \times R.$$

Note that

$$K_1 \leq \frac{4}{1-2\varepsilon E_0} K_0$$

and that t_1 depends on w^0 and E^0 only. Repeating the above procedure a finite number of times, then we have

$$0 < u(t,x), v(t,x), w(t,x) \leq \left(\frac{4}{1-2\varepsilon E_0}\right)^{(2(e-2)w^0/E_0)t} \cdot K_0$$

for any $t \in [0, +\infty)$ and then the result follows.

We next treat the case of the periodic initial data. Then we have the similar a priori estimate as follows;

Lemma 5. *Suppose that*

$$(u_0(x), v_0(x), w_0(x)) \tag{24}$$

is a periodic function with a period 2 *and that* (8) *and*

$$\int_0^2 \{u_0(x)+v_0(x)+w_0(x)\}dx = L_0 < +\infty \tag{25}$$

or suppose that (19) *and*

$$\min_{\substack{(w^0)^2=u^0v^0 \\ \delta\le u^0, v^0, w^0\le K_0}} \int_0^2 \{f(u_0(x), u^0)+f(v_0(x), v^0)+f(w_0(x), w^0)\}dx = E_0 < +\infty. \tag{26}$$

Then there exists ε_0 *such that for* $\varepsilon L_0 \le \varepsilon_0$ *or* $\varepsilon E_0 \le \varepsilon_0$ *the solution of* (C. Pr.) *is estimated by*

$$0 \le u(t,x), v(t,x), w(t,x) \le C^{Kt}K_0, \qquad (t,x)\in[0,+\infty)\times R,$$

where C *and* K *are positive constants depending on* K_0, L_0, E_0 *and* ε_0.

Proof. We can prove this lemma by using the similar procedure of Lemmas 3 and 4, noting that

$$\int_0^2 \{u(t,x)+v(t,x)+w(t,x)\}dx = L_0$$

or

$$\int_0^2 \{f(u(t,x), u^0)+f(v(t,x), v^0)+f(w(t,x), w^0)\}dx = E_0,$$

for any $t\in[0,+\infty)$.

Using Lemmas 1–5, from a standard continuation of the local solution argument, we conclude:

Theorem. *Let the initial data* {(8), (9)}, {(19), (20)}, {(8), (24), (25)} *or* {(19), (24), (26)} *be given. Then there exists* ε_0 *such that for* $\varepsilon L_0 \le \varepsilon_0$ *in* {(8), (9)} *and* {(8), (24), (25)} *and* $\varepsilon E_0 \le \varepsilon_0$ *in* {(19), (20)} *and* {(19), (24), (26)} *the Cauchy problem* (1) *and* (2) *has a non-negative global solution in* $(t,x)\in[0,+\infty)\times R$.

Remark 1. Lemma 5 assures the existence of a global solution of the mixed problem with the perfect reflective walls in the domain $[0,+\infty)\times[0,1]$, which has the non-negative initial conditions

$$\begin{aligned} u(0,x) &= u_0(x) \\ v(0,x) &= v_0(x), \qquad x\in[0,1] \\ w(0,x) &= w_0(x) \end{aligned}$$

and the boundary conditions

$$u(t,x)=v(t,x), \quad x=0 \quad \text{and} \quad x=1,\ t\in[0,+\infty),$$

where $(u_0(x), v_0(x))$ is supposed to satisfy the compatibility conditions, that is,

$$\begin{aligned} &u_0(0)=v_0(0) \\ &u_0(1)=v_0(1) \\ &\{u_0(x)\}_x|_{x=0} = -\{v_0(x)\}_x|_{x=0} \\ &\{v_0(x)\}_x|_{x=1} = -\{u_0(x)\}_x|_{x=1}. \end{aligned}$$

Remark 2. We can show a new finite difference scheme corresponding to (C. Pr.), the maximum norm stability of which is not influenced by the nonlinear term as follows;

$$u_k^n=u(n\Delta t, k\Delta x),\quad v_k^n=v(n\Delta t, k\Delta x),\quad w_k^n=w(n\Delta t, k\Delta t)$$

$$A(\Delta t, n, k)=1+\Delta t\varepsilon(u_{k-1}^n+v_{k+1}^n+2w_k^n+2K),\qquad K=\frac{2K_0}{1-\varepsilon L_0}$$

$$u_k^{n+1}=u_{k-1}^n+\Delta t\varepsilon\{(w_k^n)^2-u_{k-1}^n v_{k+1}^n\}/A(\Delta t, n, k)$$
$$v_k^{n+1}=v_{k+1}^n+\Delta t\varepsilon\{(w_k^n)^2-u_{k-1}^n v_{k+1}^n\}/A(\Delta t, n, k)$$
$$w_k^{n+1}=w_k^n-2\Delta t\varepsilon\{(w_k^n)^2-u_{k-1}^n v_{k+1}^n\}/A(\Delta t, n, k),$$

where Δt and Δx are mesh sizes in the t and x directions respectively. This scheme can be applied to the Cauchy problem (1) and {(8), (9)} or (1) and {(8), (24), (25)} and has the same a priori estimate as that of (C. Pr.).

References

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