179. On the Existence Proof of Haar Measure

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H. Cartan [1] established the existence of the Haar measure on arbitrary locally compact topological groups without the axiom of choice. In that case, the method used by him was to construct the Haar measure as a limit based on Cauchy's convergence criterion for filter. The purpose of this paper is to construct, following the idea of Cartan, the Harr measure by the method of ranked spaces. In this case, the Haar measure is given as a limit based on Cauchy's convergence criterion for "sequence".

Throughout this paper, terminologies and notations concerning ranked spaces are the same as in [2].

We begin with the definition of the ranked space $(\mathcal{K}, \mathcal{V})$ needed for our proof.

1. Definition of the ranked space $(\mathcal{K}, \mathcal{V})$. Let $\mathcal{K}$ be the vector space consisting of all continuous real-valued functions $f \geq 0$ on G with compact supports, and let $\boldsymbol{K}$ be the family of all compact subsets of G. For each $K \in \boldsymbol{K}$, we denote by $\mathcal{K}(K)$ the vector subspace of $\mathcal{K}$ formed of all functions whose supports are contained in K, and denote by $(\mathcal{K}(K), d)$ the metric space $\mathcal{K}(K)$ provided with the distance function $d(f, g) = \|f - g\|$, where $\| \ \|$ is the uniform norm. We denote the ranked union space of the ranked metric spaces $(\mathcal{K}(K), \mathcal{V}(d))$ $(K \in \boldsymbol{K})$ by $(\mathcal{K}, \mathcal{V})$.

We first know that the following holds for $(\mathcal{K}, \mathcal{V})$ by [2], Theorem 1 and Proposition 2, since $(\mathcal{K}(K), \mathcal{V}(d))$ $(K \in \boldsymbol{K})$ satisfies the condition $(\dagger_1)$ in [2].

Proposition 1. *We have* $\{r\text{-}\lim f_n\} \ni f$ *in* $(\mathcal{K}, \mathcal{V})$ *if and only if there is a* $K \in \boldsymbol{K}$ *such that* $\operatorname{supp} f \subset K$ *and* $\operatorname{supp} f_n \subset K$ $(n = 0, 1, 2, \cdots)$, *and we have* $\lim d(f_n, f) = 0$.

For $s \in G$ and $f \in \mathcal{K}$, we denote the left translation of f by $s : f(s^{-1}x)$ by sf, and the support of f by $\operatorname{supp} f$. Furthermore we denote the family of all functions f^* expressed as $f^* = \sum_{i=1}^{i_0} c_i s_i g$, where $g \in \mathcal{K}$ and $c_i \geq 0$, $s_i \in G$ $(i = 1, 2, \cdots, i_0)$, by $\mathcal{K}_0$, and denote g by $\chi(f^*)$ and $\sum_{i=1}^{i_0} c_i$ by $c(f^*)$ for such f^*.

2. Results already known (cf. [1]). We fix a function $f_0 \in \mathcal{K}$ with $f_0 \neq 0$, and for $f, \varphi \in \mathcal{K}$ with $\varphi \neq 0$, we put

$$(f : \varphi) = \inf \{c(f^*) ; f \leq f^* (\in \mathcal{K}_0), \chi(f^*) = \varphi\},$$
$$I_\varphi(f) = (f : \varphi)/(f_0 : \varphi).$$

Then $I_\varphi(f)$ is a left-invariant, non-negatively homogeneous functional on $\mathcal{K}$ satisfying

(1) $0 \leq I_\varphi(f) \leq (f : f_0)$,

(2) $|f_1 - f_2| \leq f_3$ implies $|I_\varphi(f_1) - I_\varphi(f_2)| \leq I_\varphi(f_3)$.

Moreover

Lemma 1. *If $f^* \in \mathcal{K}_0$ and $\varepsilon > 0$, there is a neighborhood V of the identity in G such that*

$$I_\varphi(f^*) \leq c(f^*) I_\varphi(\chi(f^*)) \leq I_\varphi(f^*) + \varepsilon$$

for every $\varphi \in \mathcal{K}$, $\varphi \neq 0$, with support contained in V.

As is easily seen, the theorem of approximation according to Cartan (see [1]) is given in the following form.

Theorem of approximation (Cartan). Given $f \in \mathcal{K}$ and $\varepsilon > 0$, there is a neighborhood V of the identity in G such that, for every $g \in \mathcal{K}$, $g \neq 0$, with support contained in V, there is an $f^* \in \mathcal{K}_0$ which satisfies $\chi(f^*) = g$, $d(f^*, f) < \varepsilon$ and $\operatorname{supp} f^* \subset \operatorname{supp} f$ at the same time.

3. Existence of the Haar measure. In this paper, we say that the sequences $\{f^*_{ni}; i = 0, 1, 2, \cdots\}$ $(n = 1, 2, \cdots, m)$ of points of $\mathcal{K}_0$ are of *the same type* if $\chi(f^*_{1i}) = \chi(f^*_{2i}) = \cdots = \chi(f^*_{mi})$ for every i. Then

Lemma 2. *If $f_1, f_2, \cdots, f_m \in \mathcal{K}$, there are sequences $\{f^*_{ni}; i = 0, 1, 2, \cdots\}$ $(n = 1, 2, \cdots, m)$ of points of $\mathcal{K}_0$ of the same type which r-converge to f_n $(n = 1, 2, \cdots, m)$ in $(\mathcal{K}, \mathcal{V})$ respectively.*

Proof. For each $i \in \{0, 1, 2, \cdots\}$, there are $f^*_{ni} \in \mathcal{K}_0$ $(n = 1, 2, \cdots, m)$ such that $\chi(f^*_{1i}) = \chi(f^*_{2i}) = \cdots = \chi(f^*_{mi})$ and $d(f^*_{ni}, f_n) < 1/2^i$, $\operatorname{supp} f^*_{ni} \subset \operatorname{supp} f_n$ $(n = 1, 2, \cdots, m)$, by Theorem of approximation. $\{f^*_{ni}\}$ $(n = 1, 2, \cdots, m)$ are the desired sequences by Proposition 1.

Moreover, we have

Lemma 3. *If $f \in \mathcal{K}$ and $\{f^*_n\}$, $\{h^*_n\}$ are sequences of points of $\mathcal{K}_0$ of the same type which r-converge to f, f_0 in $(\mathcal{K}, \mathcal{V})$ respectively, then there exists the limit $\lim c(f^*_n)/c(h^*_n)$, which is independent of the particular sequences $\{f^*_n\}, \{h^*_n\}$.*

Proof. Since $\{r\text{-}\lim f^*_n\} \ni f$, $\{r\text{-}\lim h^*_n\} \ni f_0$, we have, by Proposition 1, a $K \in \boldsymbol{K}$ containing the sets: $\operatorname{supp} f$, $\operatorname{supp} h$, $\operatorname{supp} f^*_n$, $\operatorname{supp} h^*_n$ $(n = 0, 1, 2, \cdots)$, and for this K, if we let k be a function $k \in \mathcal{K}$ such that $0 \leq k(x) \leq 1$ and $k(x) = 1$ if $x \in K$, we have a sequence $\{\delta_n; n = 0, 1, 2, \cdots\}$ of positive real numbers converging to 0 such that $|f(x) - f^*_n(x)| < \delta_n k(x)$ and $|f_0(x) - h^*_n(x)| < \delta_n k(x)$. Therefore, from (1), (2), Lemma 1 and that I_φ is non-negatively homogeneous, we have a sequence $\{V_n\}$ of neighborhoods of the identity in G such that

(1) $V_0 \supset V_1 \supset \cdots \supset V_n \supset \cdots$,

(2) for each n, if we put $c(f^*_n) = c_n$, $c(h^*_n) = d_n$ and $\chi(f^*_n) = \chi(h^*_n) = g_n$,

$$\begin{aligned} &|I_\varphi(f) - c_n I_\varphi(g_n)| < \delta_n(1 + (k : f_0)), \\ &|1 - d_n I_\varphi(g_n)| < \delta_n(1 + (k : f_0)) \end{aligned} \tag{1'}$$

hold for every $\varphi \in \mathcal{K}$, $\varphi \neq 0$, with support contained in V_n.

Now let us take a sequence $\{\varphi_n\}$ such that $\varphi_n \in \mathcal{K}$, $\varphi_n \neq 0$ and $\operatorname{supp} \varphi_n \subset V_n$ for each n. We then see that if $n_1 < n_2$, we have, by (1′)

$$\left|\frac{c_{n_1}}{d_{n_1}}-\frac{c_{n_2}}{d_{n_2}}\right|<\left|\frac{c_{n_1}}{d_{n_1}}-c_{n_1}I_{\varphi_{n_2}}(g_{n_1})\right|+|c_{n_1}I_{\varphi_{n_2}}(g_{n_1})-I_{\varphi_{n_2}}(f)|$$
$$+|I_{\varphi_{n_2}}(f)-c_{n_2}I_{\varphi_{n_2}}(g_{n_2})|+\left|c_{n_2}I_{\varphi_{n_2}}(g_{n_2})-\frac{c_{n_2}}{d_{n_2}}\right|$$
$$<\delta_{n_1}\left(\frac{c_{n_1}}{d_{n_1}}+\frac{c_{n_2}}{d_{n_2}}+2\right)(1+(k:f_0)).$$

On the other hand, $\{c_n/d_n\}$ is a bounded sequence. Because, without loss of generality, we may suppose $\delta_n(1+(k:f_0))<1/2$ for every n, and so by (1′),

$$c_n/d_n=c_nI_{\varphi_n}(g_n)/d_nI_{\varphi_n}(g_n)<2(f:f_0)+1.$$

Since, therefore, $\{c_n/d_n\}$ is a Cauchy sequence, there exists the limit $\lim c_n/d_n$. Moreover we have

$$\lim c_n/d_n=\lim I_{\varphi_n}(f). \tag{2′}$$

The same argument is applied to the other $\{f_n^{**}\}, \{h_n^{**}\}$. In this case we may have a $\{\varphi_n\}$ for which (2′) simultaneously hold for $\{f_n^*\}, \{h_n^*\}$ and $\{f_n^{**}\}, \{h_n^{**}\}$. Hence $\lim c(f_n^*)/c(h_n^*)=\lim c(f_n^{**})/c(h_n^{**})$ follows.

Thus, by Lemmas 2 and 3, we can define $I(f)$ for every $f \in \mathcal{K}$ as follows:

$$I(f)=\lim c(f_n^*)/c(h_n^*),$$

where $\{f_n^*\}, \{h_n^*\}$ are sequences of points of $\mathcal{K}_0$ of the same type which r-converge to f, f_0 in $(\mathcal{K}, \mathcal{V})$ respectively.

Then, we see that

Proposition 2. *$I(f)$ is a left-invariant positive linear functional on $\mathcal{K}$.*

Proof. $I(sf)=I(f)$ $(s \in G)$ and $I(af)=aI(f)$ $(a \geq 0)$ follow from that, in general, if $\{r\text{-}\lim k_n^*\} \ni k$ $(k_n^* \in \mathcal{K}_0, k \in \mathcal{K})$, then $\{r\text{-}\lim sk_n^*\} \ni sk$, $\{r\text{-}\lim ak_n^*\} \ni ak$, $\chi(sk_n^*)=\chi(k_n^*)$, $\chi(ak_n^*)=\chi(k_n^*)$ and $c(sk_n^*)=c(k_n^*)$, $c(ak_n^*)=a(c(k_n^*))$. Similarly, $I(f_1+f_2)=I(f_1)+I(f_2)$ follows from that, by Lemma 2, there are sequences $\{f_{1n}^*\}$, $\{f_{2n}^*\}$ and $\{h_n^*\}$ of points of $\mathcal{K}_0$ of the same type which r-converge to f_1, f_2 and f_0 in $(\mathcal{K}, \mathcal{V})$ respectively.

By Proposition 2, $I(f)$ furnishes a Haar's left-invariant measure on G for which the measure of f_0 is equal to 1.

4. Uniqueness of the Haar measure. Let $J(f)$ be the other Haar's left-invariant measure on G, and let $\{f_n^*\}$, $\{h_n^*\}$ be sequences of points of $\mathcal{K}_0$ of the same type r-converging to f, f_0 in $(\mathcal{K}, \mathcal{V})$ respectively. Since then, by Proposition 1 and [2], Proposition 3, $J(f)$ is continuous on $(\mathcal{K}, \mathcal{V})$ in the sense of [2], 5, we have $\lim J(f_n^*)=J(f)$ and $\lim J(h_n^*)=J(f_0)$. Therefore, it follows that

$$\frac{J(f)}{J(f_0)}=\lim\frac{J(f_n^*)}{J(h_n^*)}=\lim\frac{c(f_n^*)}{c(h_n^*)}=I(f).$$

References

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[2] S. Nakanishi: On the strict union of ranked metric spaces. Proc. Japan Acad., **50**, 603–607 (1974).