1. *Finiteness of Objects in Categories*

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Introduction. The well-known notion of ultraproducts in model theory was re-defined in terms of categories in [2], and it was observed that having an injective diagonal map $d: B \to B^{\Lambda}/D$ for any set Λ and its ultrafilter D is essential for the object B to have an algebraic (finitary) structure. However, there, we dealt with only concrete categories. When we deal with sets such as objects in concrete categories, we tacitly assume that the notion of finiteness is well understood, but in order to generalize the theorems into abstract categories, we need to define the notion of finiteness in terms of categories.

We made some attempt in [3] to describe the finiteness of structure in objects in terms of categories, and thereby some of theorems in [2] were generalized. Here, we make another attempt to describe the finiteness of objects themselves, and the theorems in [2] which were omitted of discussion in [3] will be wholly generalized. Theorem 1, Theorem 2 and Theorem 3 below correspond to Lemma 9, Lemma 10 and Theorem 8 in [2] respectively.

As for the definitions of terms such as *compatible family of morphisms*, *finitary objects* and *ultraproducts*, refer to [3], and for more basic terms of categories, to Isbell [1].

§ **1.** Let $\mathfrak{C}$ be an abstract locally small category which is complete to the both sides, $Ob(\mathfrak{C})$ the collection of all its objects, and for A, $B \in Ob(\mathfrak{C})$, $\mathfrak{C}(A, B)$ the set of all morphisms from A to B.

Definition. For objects G and B, G is said to *separate* B, if for any coterminal morphisms $f, f': B \rightrightarrows B'$ such that $f \neq f'$, there exists an $s: G \to B$ such that $fs \neq f's$. An object B is called *finite*, if there exists a $G \in Ob(\mathfrak{C})$ such that G separates all powers of B and $\mathfrak{C}(G, B)$ consists of only finite number of morphisms. G is said to *represent the finiteness of* B.

Theorem 1. *If an object B is finite, then the diagonal map $d: B \to B^{\Lambda}/D$ to an ultrapower is epimorphic for any set Λ and its ultrafilter D.*

Proof.

$$
\begin{tikzcd}
G \arrow[r, "b"] \arrow[d, "h"'] \arrow[dr, "h'"'] & B \arrow[d, shift right, "d_{\Xi_b}"'] \arrow[drr, "d"] & & \\
B^{\Xi} \arrow[r, "\pi^{\Xi}_{\Xi_b}"'] & B^{\Xi_b} \arrow[u, shift right, "\pi^{\Xi_b}_{\xi}"'] \arrow[r, "\pi^{\Xi_b}"'] & B^A/D \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X
\end{tikzcd}
$$

Assume $f, f' : B^A/D \rightrightarrows X$ and $f \neq f'$. Since B^A/D is the direct limit of the product system over D, there exists a $\Xi \in D$ such that $f\pi^{\Xi} \neq f'\pi^{\Xi}$ where $\pi^{\Xi} : B^{\Xi} \to B^A/D$ is the canonical injection. Let G be the object that represents the finiteness of B. Then there exists an $h : G \to B$ such that $f\pi^{\Xi}h \neq f'\pi^{\Xi}h$. For $\xi \in \Xi$, let $\pi^{\Xi}_{\xi} : B^{\Xi} \to B_{\xi}(=B)$ be the canonical projection and for $b \in \mathfrak{C}(G, B)$, put $\Xi_b = \{\xi \in \Xi \mid \pi^{\Xi}_{\xi} h = b\}$. Then since $\mathfrak{C}(G, B)$ is finite, there exists one and only one $b \in \mathfrak{C}(G, B)$ such that $\Xi_b \in D$. Let $\pi^{\Xi}_{\Xi_b} : B^{\Xi} \to B^{\Xi_b}$ be the projection, and put $h' = \pi^{\Xi}_{\Xi_b} h$. Then for every $\xi \in \Xi_b$, we have $\pi^{\Xi_b}_{\xi} h' = b$ and $f\pi^{\Xi_b}h' \neq f'\pi^{\Xi_b}h'$. Hence $h' = d^{\Xi_b}b$, where $d_{\Xi} : B \to B^{\Xi}$ is the diagonal morphism such that $\pi^{\Xi}_{\xi} d_{\Xi} = 1_{B_{\xi}}$ for every $\xi \in \Xi$, and we have $f\pi^{\Xi_b}d_{\Xi_b}b \neq f'\pi^{\Xi_b}d_{\Xi_b}b$. Since $d = \pi^{\Xi}d_{\Xi}$ for $\Xi \in D$, we have $fdb \neq f'db$, and hence $fd \neq f'd$. q.e.d.

Theorem 2. *If B is finitary and finite, then the diagonal map $d : B \to B^A/D$ is an isomorphism.*

Proof.

$$
\begin{tikzcd}
 & G \arrow[dl, "b"'] \arrow[d, "g"] \arrow[dr, "g'"] & & \\
B \arrow[r, "d_{\Xi}"'] \arrow[dr, equal, "1_B"'] & B^{\Xi} \arrow[r, "\pi^{\Xi}_{\Xi'}"'] \arrow[d, "e^{\Xi}"] & B^{\Xi'} \arrow[r, "\pi^{\Xi'}"] \arrow[dl, "e^{\Xi'}"] & B^A/D \arrow[dll, shift left, "e"] \\
 & B \arrow[urr, shift left, "d"] & &
\end{tikzcd}
$$

Let G be an object that represents the finiteness of B, and assume $\Xi \in D$. Since G separates B^{Ξ}, a morphism $Y \to B^{\Xi}$ that divides all morphisms in $\mathfrak{C}(G, B^{\Xi})$ must be epimorphic. This means that $\mathfrak{C}(G, B^{\Xi})$ covers B^{Ξ} (cf. [3]). For each $g : G \to B^{\Xi}$ there exists one and only one $b \in \mathfrak{C}(G, B)$ such that the set $\Xi_b(g) = \{\xi \in \Xi \mid \pi^{\Xi}_{\xi} g = b\}$ belongs to D. Put $\varphi_{\Xi}(g) = b$. Then the set of pairs $\{(g, \varphi_{\Xi}(g)) \mid g \in \mathfrak{C}(G, B^{\Xi})\}$ is finitely compatible. Indeed, for $g_1, g_2, \cdots, g_n \in \mathfrak{C}(G, B^{\Xi})$ the set $\Xi_0 = \bigcap_{k=1}^{n} \Xi_{b_k}(g_k)$ is not void, and for $\xi \in \Xi_0$ we have $\pi^{\Xi}_{\xi} g_k = \varphi_{\Xi}(g_k)$ for $k = 1, 2, \cdots, n$. Since $\mathfrak{C}(G, B^{\Xi})$ covers B^{Ξ} and B is finitary, there exists an $e^{\Xi} : B^{\Xi} \to B$ such that $e^{\Xi}g = \varphi_{\Xi}(g)$ for every $g \in \mathfrak{C}(G, B^{\Xi})$. In general, if $\Xi, \Xi' \in D$ and

$\Xi'\subset\Xi$, then, putting $g'=\pi^{\Xi}_{\Xi'}g$, we have $\pi^{\Xi'}_{\xi}g'=\pi^{\Xi}_{\xi}g$ for $\xi\in\Xi'$. Hence the $b\in\mathfrak{C}(G,B)$ such that $\Xi_b(g)\in D$ is the same as the one such that $\Xi'_b(g')\in D$. Hence $e^{\Xi}g=e^{\Xi'}g'=e^{\Xi'}\pi^{\Xi}_{\Xi'}g$ for all $g: G\to B^{\Xi}$. Since G separates B^{Ξ}, we have $e^{\Xi}=e^{\Xi'}\pi^{\Xi}_{\Xi'}$. Now particularly, for $b\in\mathfrak{C}(G,B)$ we have $\varphi_{\Xi}(d_{\Xi}b)=b$, i.e., $e^{\Xi}d_{\Xi}b=b$. Again, since $\mathfrak{C}(G,B)$ covers B, we have $e^{\Xi}d_{\Xi}=1_B$ for all $\Xi\in D$. Now $e^{\Xi}: B^{\Xi}\to B$ induces an $e: B^A/D\to B$ such that $e^{\Xi}=e\pi^{\Xi}$ for all $\Xi\in D$. Thus we have $1_B=e^{\Xi}d_{\Xi}=e\pi^{\Xi}d_{\Xi}=ed$ and d is a left reversible epimorphism, and hence an isomorphism.

q.e.d.

Lemma. *If B is finitary and finite, then for any morphism $f: A\to B$ and an ultrapower A^A/D of A, there exists a $g: A^A/D\to B$ such that $f=gd$ where $d: A\to A^A/D$ is the diagonal map.*

Proof. In general a morphism $f: A\to B$ naturally induces a morphism $f^{\Xi}: A^{\Xi}\to B^{\Xi}$ for any index set Ξ, and hence a morphism $g: A^A/D\to B^A/D$ with which we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "d_{\Xi}"] \arrow[d, "f"'] & A^{\Xi} \arrow[r, "\pi^{\Xi}"] \arrow[d, "f^{\Xi}"'] & A^A/D \arrow[d, "g"'] \\
B \arrow[r] & B^{\Xi} \arrow[r] & B^A/D .
\end{tikzcd}$$

Particularly, if B is finitary and finite, then the row $B\to B^{\Xi}\to B^A/D$ is an isomorphism. Hence $f=g\pi^{\Xi}d_{\Xi}=gd$. q.e.d.

Theorem 3. *If A is strongly finitary* (*cf.* [3]), *then the diagonal map $d: A\to A^A/D$ is an extremal monomorphism.*

Proof.

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "e"'] & A^A/D \arrow[dl, "g"'] \arrow[d, "g_{\nu}"] \\
\Pi_{\nu}B_{\nu} \arrow[r, "p_{\nu}"] & B_{\nu}
\end{tikzcd}$$

Since A is strongly finitary, it is an extremal subobject $e: A\to\Pi_{\nu}B_{\nu}$ of a direct product, of which the components B_{ν} are all finitary and finite. Letting $p_{\nu}: \Pi_{\nu}B_{\nu}\to B_{\nu}$ be the canonical projection, each $p_{\nu}e: A\to B_{\nu}$ determines a g_{ν} $A^A/D\to B_{\nu}$ such that $p_{\nu}e=g_{\nu}d$ by the lemma above. Now those g_{ν} determines a $g: A^A/D\to\Pi_{\nu}B_{\nu}$ such that $p_{\nu}g=g_{\nu}$ for all ν. Now $p_{\nu}gd=g_{\nu}d=p_{\nu}e$ for all ν and hence $gd=e$. Thus d is an initial factor of an extremal monomorphism e, and hence an extremal monomorphism itself. q.e.d.

Now all theorems discussed in [2] for concrete categories were generalized into abstract categories. Here we add a theorem that shows some properties naturally expected for the notion of finiteness.

Theorem 4. *Let B be a finite object. Then*

(i) *$\mathfrak{C}(B,B)$ consists of a finite number of morphisms, and*

(ii) *if $m: B \to B$ is monomorphic, then it is epimorphic, and vice versa.*

Proof. Let G be an object that represents the finiteness of B.

(i) We shall show that if $\mathfrak{C}(G,B)$ consists of n morphisms, then $\mathfrak{C}(B,B)$ contains n^n morphisms at most. Indeed, by the principal covariant representation h^G (cf. [1]), each $f: B \to B$ induces an endomorphism $h^G(f): \mathfrak{C}(G,B) \to \mathfrak{C}(G,B)$. Since there are at most n^n endomorphisms of $\mathfrak{C}(G,B)$, if $\mathfrak{C}(B,B)$ contains more than n^n morphisms, there must be two $f, f' \in \mathfrak{C}(B,B)$ with $f \neq f'$ such that $h^G(f) = h^G(f')$, that is, $fg = f'g$ for every $g \in \mathfrak{C}(G,B)$. This contradicts that G separates B.

(ii) Assume that $m: B \to B$ is monomorphic. Then its covariant principal representation by G, $h^G(m): \mathfrak{C}(G,B) \to \mathfrak{C}(G,B)$, is one-to-one. Since $\mathfrak{C}(G,B)$ is finite, it is also onto. If $u, v: B \to X$ and $u \neq v$, then, since G separates B, there exists an $f: G \to B$ such that $uf \neq vf$. Since $h^G(m)$ is onto, there exists a $g: G \to B$ such that $f = mg$. Now we have $umg \neq vmg$ and hence $um \neq vm$. The converse can be proved similarly.

q.e.d.

References

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