23. The Fundamental Solution for a Parabolic Pseudo-Differential Operator and Parametrices for Degenerate Operators

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(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1975)

Introduction. In the present paper we shall construct the fundamental solution $E(t, s)$ for a parabolic pseudo-differential equation

$$(0.1)\qquad \begin{cases} Lu=\dfrac{\partial u}{\partial t}+p(t\,;x,D_x)u=0 & \text{in } (0,\infty)\times R^n \\ u|_{t=0}=u_0 \end{cases}$$

where $p(t\,;x,D_x)$ is a pseudo-differential operator of class $\mathcal{E}_t^0(S^m_{\lambda,\rho,\delta})$ $(0\leqslant\rho\leqslant 1,\ -\infty<\delta<1,\ \delta<\rho)$ which satisfies the following condition:

There exist positive constants C_0 and R such that

(0.2) $\operatorname{Re} p(t\,;x,\xi)\geqslant C_0\lambda(x,\xi)^m$ for $0\leqslant t<\infty$ and $|x|+|\xi|\geqslant R$,

where $\lambda=\lambda(x,\xi)$ is a basic weight function defined in § 1. We note that $\lambda(x,\xi)$ varies even in x and may increase in polynomial order, and that it is important to take $\delta<0$ in § 4.

The fundamental solution $E(t,s)$ will be constructed as a pseudo-differential operator of class $S^0_{\lambda,\rho,\delta}$ with parameter t and s. The method of construction of $E(t,s)$ is similar to that given in Tsutsumi [10]. Then the solution of the Cauchy problem (0.1) is given by $u(t)=E(t,0)u_0$.

In § 3 we show that if $P(t)$ is a positive operator, then $\exp\{c(t-s_0)E(t,s_0)\}$ are bounded in $S^{-N}_{\lambda,\rho,\delta}$ for $t\geqslant t_0>s_0\geqslant 0$, where c is a positive constant and N is any number.

As an application of the above theorems, in § 4 we construct the fundamental solution $E_0(t)$ for a degenerate parabolic operator

$$(0.3)\qquad L_0=\frac{\partial}{\partial t}+D_x^{2l}+x^{2k}D_y^{2m}=\frac{\partial}{\partial t}+P_0$$

and apply $E_0(t)$ to construct the parametrix for P_0 near $x=0$ in some class of pseudo-differential operator. We note that in case $l=k=m=1$ the precise symbol of the fundamental solution $E_0(t)$ is found in Hoel [4] and that the operator P_0 has been studied by Beals [1], Hörmander [3], Grushin [2], Kumano-go and Taniguchi [6] and Sjöstrand [9].

§ 1. Notations and basic calculus of pseudo-differential operators of class $S^m_{\lambda,\rho,\delta}$. We say that a C^∞-function $\lambda(x,\xi)$ in $R^n_x\times R^n_\xi$ is a basic weight function when $\lambda(x,\xi)$ satisfies conditions (cf. [6]):

(i) $A^{-1}(1+|x|+|\xi|)^a \leqslant \lambda(x,\xi) \leqslant A(1+|x|^{\tau_0}+|\xi|)$ $(a \geqslant 0, \tau_0 \geqslant 0, A>0)$.

(ii) $|\lambda_{(\beta)}^{(\alpha)}(x,\xi)| \leq A_{\alpha,\beta}\lambda(x,\xi)^{1-|\alpha|+\delta|\beta|}$

$(0 \leqslant \rho \leqslant 1, -\infty < \delta < 1, \delta < \rho, A_{\alpha,\beta}>0)$ for any α, β.

(iii) $\lambda(x+y,\xi) \leqslant A_1(1+|y|)^{\tau_1}\lambda(x,\xi)$ $(\tau_1 \geqslant 0, A_1>0)$,

where

$$\lambda_{(\beta)}^{(\alpha)}(x,\xi)=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}(-i\partial/\partial x_1)^{\beta_1}\cdots(-i\partial/\partial x_n)^{\beta_n}\lambda(x,\xi),$$
$$|\alpha|=\alpha_1+\cdots+\alpha_n, \ |\beta|=\beta_1+\cdots+\beta_n$$

for any multi index $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)$.

We denote by $S^m_{\lambda,\rho,\delta}(-\infty<m<\infty, 0 \leqslant \rho \leqslant 1, -\infty<\delta<1, \delta<\rho)$ the set of all C^∞-symbols $p(x,\xi)$ defined in $R^n_x \times R^n_\xi$ which satisfies for any α, β

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \leqslant C_{\alpha,\beta}\lambda(x,\xi)^{m-\rho|\alpha|+\delta|\beta|}$$

for some constant $C_{\alpha,\beta}$. For a symbol $p(x,\xi) \in S^m_{\lambda,\rho,\delta}$ we define a pseudo-differential operator by

$$Pu(x)=p(x,D_x)u(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi,$$

where $d\xi=(2\pi)^{-n}d\xi$ and $\hat{u}(\xi)$ denote the Fourier transform of $u(x)$ in $\mathcal{S}$ defined by

$$\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx.$$

For $p(x,\xi) \in S^m_{\lambda,\rho,\delta}$ we define semi-norms $|p|_l^{(m)}$, $l=0,1,\cdots$ by

$$|p|_l^{(m)}=\max_{|\alpha|+|\beta|\leqslant l}\left\{\sup_{(x,\xi)}|p_{(\beta)}^{(\alpha)}(x,\xi)|\,\lambda(x,\xi)^{-m+\rho|\alpha|-\delta|\beta|}\right\}.$$

Then $S^m_{\lambda,\rho,\delta}$ makes a Fréchet space. Set $S_\lambda^{-\infty}=\bigcap_{-\infty<m<-\infty} S^m_{\lambda,\rho,\delta}$.

Theorem 1.1. *Let $P_j=p_j(x,D_x) \in S^{m_j}_{\lambda,\rho,\delta}$ $(j=1,2,\cdots,\nu)$. Then $P=P_1P_2\cdots P_\nu$ belongs to $S^m_{\lambda,\rho,\delta}$, where $m=\sum_{j=1}^\nu m_j$. Moreover for any positive integer l, there exist C_1 and $\tilde{l}$ such that*

$$|\sigma(P)|_l^{(m)} \leqslant C_1^\nu \prod_{j=1}^\nu |p_j|_{\tilde{l}}^{(m_j)}$$

where $\tilde{l}$ depends on $M=\sum_{j=1}^\nu |m_j|<\infty$ and l but is independent of ν.

From the above theorem the following theorem is proved by the same method in Kumano-go [5].

Theorem 1.2. *Let $P \in S^0_{\lambda,\rho,\delta}$. Then there exists l such that*

$$\|Pu\| \leqslant C_2|p|_l^{(0)}\|u\| \qquad \text{for any } u \in L^2,$$

where $\|\cdot\|$ is the $L^2(R^n)$ norm.

For any $s>0$ we define $H_{\lambda,s}$ by $H_{\lambda,s}=\{u \in L^2; \lambda^s(x,D_x)u \in L^2\}$ with the norm $\|u\|^2_{\lambda,s}=\{\|\lambda^s(x,D_x)u\|^2+\|u\|^2\}$.

If the basic weight function $\lambda(x,\xi)$ satisfies (i) for $a>0$, then we get by Theorem 1.2.

Proposition. *Let $0 \leqslant s_1 < s_2$. Then for any $\varepsilon>0$ there is a positive constant C_ε such that*

$$\|u\|_{\lambda,s_1} \leqslant \varepsilon\|u\|_{\lambda,s_2}+C_\varepsilon\|u\|.$$

We get the expansion formula as follows.

Theorem 1.3 (cf. [6]). *Let $P_j \in S^{m_j}_{\lambda,\rho,\delta}$ $(j=1,2)$. Then we have the expansion for any N*

$$\sigma(P_1P_2)(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}p_1^{(\alpha)}(x,\xi)p_{2(\alpha)}(x,\xi)+r_N(x,\xi),$$

where $r_N(x,\xi)\in S_{\lambda,\rho,\delta}^{m_1+m_2-(\rho-\delta)N}$.

§ 2. Construction of fundamental solution. Definition 2.1 (cf. [10]). $\mathcal{E}_t^0(S_{\lambda,\rho,\delta}^m)(\mathcal{E}_t^\infty(S_{\lambda,\rho,\delta}^m))$ is the set of all functions $p(t;x,\xi)$ of class $S_{\lambda,\rho,\delta}^m$ which are continuous (infinitely differentiable) with respect to parameter t for $t\geqslant 0$.

Definition 2.2 (cf. [10]). We say $\{p_j(x,\xi)\}_{j=0}^\infty$ of $S_{\lambda,\rho,\delta}^m$ converges to $p(x,\xi)\in S_{\lambda,\rho,\delta}^m$ weakly, if $\{p_j(x,\xi)\}_{j=0}^\infty$ make a bounded set of $S_{\lambda,\rho,\delta}^m$ and $p_{j(\beta)}^{(\alpha)}(x,\xi)$ converges to $p_{(\beta)}^{(\alpha)}(x,\xi)$ as $j\to\infty$ uniformly on K for any α,β, where K is any compact set in $R_x^n\times R_\xi^n$. We denote by $w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^m)$ the set of all functions $p(t,s;x,\xi)$ of class $S_{\lambda,\rho,\delta}^m(0\leqslant s\leqslant t)$ which are continuous with respect to parameters t and s with weak topology of $S_{\lambda,\rho,\delta}^m$.

Theorem 2.1. *Under the assumption* (0.2) *we can construct* $E(t,s)=e(t,s;x,D_x)\in w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^0)(0\leqslant s\leqslant t)$ *which satisfies the following properties*:

(i) $LE(t,s)=0$ *in* $t>s$.

(ii) $E(s,s)=I$.

(iii) *For any* N *such that* $-N(\rho-\delta)+m\leqslant 0$ *we can write*

$$e(t,s;x,\xi)=\sum_{j=0}^{N-1}e_j(t,s;x,\xi)+r_N(t,s;x,\xi),$$

where

$$e_0(t,s;x,\xi)=\exp\left[-\int_s^t p(\sigma;x,\xi)d\sigma\right],\qquad e_j(t,s;x,\xi)\in w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^{-(\rho-\delta)j})$$

and $r_N(t,s;x,\xi)\in w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^{-(\rho-\delta)N+m})$. *Moreover we get*

$$e_{j(\beta)}^{(\alpha)}(t,s;x,\xi)=a_{j,\alpha,\beta}(t,s;x,\xi)e_0(t,s;x,\xi)\qquad(j\geqslant 1),$$

where

$$|a_{j,\alpha,\beta}(t,s;x,\xi)|\leqslant C'_{\alpha,\beta}\lambda(x,\xi)^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j}\sum_{k=2}^{|\alpha|+|\beta|+2j}\left\{\int_s^t \mathrm{Re}\,p(\sigma;x,\xi)d\sigma\right\}^k.$$

Also, $E(t,s)$ *is unique in class* $w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^k)$ *satisfying* (i) *and* (ii) *for any* k.

We can construct $E(t,s)$ by the same method with the proof of Theorem in [10], using Theorem 1.1 and Theorem 1.3. The uniqueness is proved applying the energy inequality.

Example 1. $L_1=\dfrac{\partial}{\partial t}+D_x^{2l}+x^{2k}$ in $(0,\infty)\times R_x^1$.

Example 2. $L_2=\dfrac{\partial}{\partial t}+(D_x+ix^k)(D_x-ix^k)$ in $(0,\infty)\times R_x^1$.

We can take $\lambda(x,\xi)=(1+\xi^{2l}+x^{2k})^{1/2l}$, $\rho=1$, $\delta=-l/k$, $m=2l$ in Example 1 and $\lambda(x,\xi)=(1+\xi^2+x^{2k})^{1/2}$, $\rho=1$, $\delta=-1/k$, $m=2$ in Example 2.

Theorem 2.2. *Under the same condition with Theorem* 2.1 *the adjoint operator* $E^*(t,s)$ $(\in w-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^0))$ *satisfies*

$$\begin{cases} \dfrac{\partial}{\partial t}E^*(t,s)+E^*(t,s)P^*(t)=0 & \text{in } t>s, \\ E^*(s,s)=I \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial s}E^*(t,s)+P^*(s)E^*(t,s)=0 & \text{in } t>s, \\ E^*(t,t)=I. \end{cases}$$

Corollary. *If $P(t)$ is independent of t, then the fundamental solution $E(t,s)=E(t-s)$ satisfies also*

$$\frac{\partial}{\partial t}E(t)+E(t)P=0 \qquad \text{in } t>0.$$

If $P=P^*$, then $E(t)=E^*(t)$.

Remark. We can prove the similar theorems in this section for $p(t;x,\xi)\in\mathcal{E}_t^0(S^m_{\lambda,\rho,\delta})$ under the conditions

$$\begin{cases} \operatorname{Re} p(t;x,\xi)\geqslant c_0\lambda(x,\xi)^{m'} & 0\leqslant m'\leqslant m, \\ |p^{(\alpha)}_{(\beta)}(t;x,\xi)/\operatorname{Re}p\,(t;x,\xi)|\leqslant C_{\alpha,\beta}\lambda(x,\xi)^{-\rho|\alpha|+\delta|\beta|} & \text{for any } \alpha,\beta \end{cases}$$

by using complex powers $\{P_z(x,D_x)\}$ for $P(x,D_x)$ (cf. [7], [10]).

§ 3. **Behavior of $E(t,s)$ at $(t-s)\to\infty$.** In this section let $p(t;x,\xi)\in\mathcal{E}_t^\infty(S^m_{\lambda,\rho,\delta})(m>0)$ satisfy (0.2) and

(3.1) $\qquad \operatorname{Re}(P(t)u,u)\geqslant c_1\|u\|^2,\quad 0\leqslant t<\infty \quad$ for any $u\in\mathcal{S}$,

with a positive constant c_1. Moreover let the basic weight function $\lambda(x,\xi)$ satisfy (i) for $a>0$.

Theorem 3.1. *Let $t_0>s_0\geqslant 0$. Then for any integers l_j ($j=1,2,3$) there exists a positive constant $C(l_j,t_0,s_0)$ such that*

$$|\partial_t^{l_1}e(t,s_0)|_{l_3}^{(-l_2)}\leqslant C(l_j,t_0,s_0)\exp\{-c_2(t-t_0)\} \qquad \textit{for } t\geqslant t_0$$

where c_2 is any number $c_2<c_1$.

Note that $e(t,s;x,\xi)\in w-\mathcal{E}^\infty_{t,s}(S_\lambda^{-\infty})(t>s)$ according to Theorem 2.1, and that $f(t,s;x,\xi)=e^{ix\cdot\xi}e(t,s;x,\xi)$ satisfies

$$Lf(t,s;x,\xi)=0 \qquad \text{in } t>s.$$

Then Theorem 3.1 is proved by the following lemmas.

Lemma 3.1. *Let $u(t)\in\mathcal{E}_t^\infty(\mathcal{S})$ satisfy $Lu(t)=g(t)$ in $t>t_0$. Then for any $b\geqslant 0$ and any $c_2<c_1$ there exists $B>0$ such that*

$$\|u(t)\|_{\lambda,b}\leqslant B\left[\exp\{-c_2(t-t_0)\}\|u(t_0)\|_{\lambda,b}+\int_{t_0}^t\exp\{-c_2(t-\sigma)\}\|g(\sigma)\|_{\lambda,b}d\sigma\right].$$

Lemma 3.2. *For any $u\in\mathcal{S}$*

$$C_b^{-1}|u|_{[ab-(n+1)/2],\mathcal{S}}\leqslant\|u\|_{\lambda,b}\leqslant C_b|u|_{[\tilde{\tau}_0(b+1)+(n+1)/2],\mathcal{S}},$$

where $|u|_{b,\mathcal{S}}=\sup_{|\alpha|+|\beta|\leqslant b}|(1+|x|)^\alpha\partial_x^\beta u(x)|$ and $\tilde{\tau}_0=\max(1,\tau_0)$.

§ 4. **Application to operators of degenerate type.** At first we apply the above theorems for the construction of fundamental solution for L_0. If we construct the fundamental solution $f(t;x,D_x,\eta)$ for $(\partial/\partial t)+D_x^{2l}+x^{2k}\eta^{2m}$, then $f(t;x,D_x,D_y)$ is the fundamental solution for L_0. $f(t;x,\xi,\eta)$ is given by

$$(4.1)\qquad \begin{aligned} f(t;x,\xi,\eta)&=e(t|\eta|^{\sigma};x|\eta|^{\sigma/2l},\xi|\eta|^{-\sigma/2l}) && (\eta\neq 0),\\ &=\exp(-\xi^{2l}t) && (\eta=0),\end{aligned}$$

where $\sigma=2lm/(k+l)$ and $e(t;x,\xi)$ is the symbol of the fundamental solution of L_1 of Example 1. With respect to $f(t;x,\xi,\eta)$, we get by Theorem 2.1, Theorem 3.1 and (4.1)

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\eta}^{\gamma}f(t;x,\xi,\eta)|\leqslant C_{\alpha,\beta,\gamma}\mu(x,\xi,\eta)^{-\beta-(l/k)\alpha}|\eta|^{(m/k)\alpha-\gamma}\qquad \eta\neq 0$$

and $f(t;x,\xi,\eta)\in S_{\lambda}^{-\infty}(t>0)$, where $\mu(x,\xi,\eta)=|\xi|+|x|^{k/l}|\eta|^{m/l}+|\eta|^{m/(k+l)}$.

Set

$$\int_0^{\infty}f(t;x,\xi,\eta)dt=k(x,\xi,\eta).$$

Then from Theorem 2.1 and Theorem 3.1 we have

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\eta}^{\gamma}k(x,\xi,\eta)|\leqslant C_{\alpha,\beta,\gamma}\mu(x,\xi,\eta)^{-2l-\beta-(l/k)\alpha}|\eta|^{(m/k)\alpha-\gamma}\qquad \eta\neq 0.$$

A left and right parametrix Q for P_0 is constructed by using $k(x,\xi,\eta)$ for $|\xi|\leqslant c|\eta|^{m/l}$ and the usual method of construction of the parametrix for $|\xi|\geqslant c|\eta|^{m/l}$. $\sigma(Q)=q(x,\xi,\eta)$ satisfies

$$(4.2)\qquad |\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\eta}^{\gamma}q(x,\xi,\eta)|\leqslant C_{\alpha,\beta,\gamma}\tilde{\mu}(x,\xi,\eta)^{-2l-\beta-(l/k)\alpha}\nu(\xi,\eta)^{(m/k)\alpha-\gamma}\quad \text{for any } \xi,\eta,$$

where $\nu(\xi,\eta)=1+|\xi|^{l/m}+|\eta|$ and $\tilde{\mu}(x,\xi,\eta)=1+\mu(x,\xi,\eta)$.

We note that $q(x,\xi,\eta)$ belongs to $S_{\Phi,\varphi}^{-2l\log\tilde{\mu}}$ treated in Beals [1], if we choose weight vector $\Phi_1=\tilde{\mu}^{l/k}\nu^{m(k-l)/k(k+l)}$, $\Phi_2=\nu$, $\varphi_1=\tilde{\mu}^{l/k}\nu^{-m/k}$, $\varphi_2=1$ in case $k\geqslant l$ and $\Phi_1=\tilde{\mu}$, $\Phi_2=\nu$, $\varphi_1=\tilde{\mu}\nu^{-2m/(k+l)}$, $\varphi_2=1$ in case $k<l$ by (4.2).

Let $P=D_x-ix^kD_y^m$ (cf. [6], [8]). We consider $(\partial/\partial t)+P^*P$ and $(\partial/\partial t)+PP^*$ applying the similar argument. Then we get that P has a left parametrix if $k=$even and a right parametrix if $k=$even or $k=$odd and $m=$even.

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