21. *On the Boundedness of Integral Transformations with Highly Oscillatory Kernels*

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§ 1. **Preliminaries.** The aim of this note is to prove the $L^2(R^n)$ boundedness of a class of integral transformations which play a fundamental rôle in our notes [2] and [3].

§ 2. **Assumptions.** We shall treat the following integral transformation;

$$Af(x)=\int_{R^n} a(x, y) \exp (i\lambda S(x, y))f(y)dy, \qquad \lambda>0, \tag{1}$$

which is defined at least for $f \in C_0^\infty(R^n)$. Let $|x|$ denote the length of n vector x. Our assumptions are the following;

(A-I) $S(x, y)$ is a real infinitely differentiable function on $R^n \times R^n$.

(A-II) $\Phi=|\text{grad}_x (S(x, y)-S(x, z))|\geqq \Xi_1(x, y, z)\theta(|y-z|)$,

$\Psi=|\text{grad}_y (S(x, y)-S(z, y))|\geqq \Xi_2(x, y, z)\theta(|x-z|)$,

where $\Xi_1(x, y, z)>\delta>0$, $\Xi_2(x, y, z)>\delta>0$, and $\theta(t)=(10\sqrt{n})^{\sigma-1}t$ for $0<t<10\sqrt{n}$ and $=t^\sigma$ for $10\sqrt{n}<t$.

(A-III) For any multi-index α there exists a constant $C>0$ such that we have

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha (S(x, y)-S(x, z))\right|\leqq C\Phi$$

$$\left|\left(\frac{\partial}{\partial y}\right)^\alpha (S(x, y)-S(z, y))\right|\leqq C\Psi.$$

(A-IV) For any multi-index α there exists a constant $C>0$ such that we have

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha (a(x, y)a(x, z))\right|\leqq C\Xi_1(x, y, z)^{|\alpha|}$$

$$\left|\left(\frac{\partial}{\partial y}\right)^\alpha (a(x, y)a(z, y))\right|\leqq C\Xi_2(x, y, z)^{|\alpha|}.$$

§ 3. **Result.** Let $\|f\|$ denote the usual L^2 norm of a function f.

Theorem. *If assumptions* (A-I), (A-II), (A-III) *and* (A-IV) *hold, we have estimate*

$$\|Af\|\leqq C\lambda^{-n/2}\|f\|, \qquad \text{for } \lambda>1.$$

Here C is a positive constant independent of λ and f.

§ 4. **Proof.** Let $g_0=0, g_1, g_2, \cdots, g_k, \cdots$ be unit lattice points of R^n. Let $\{\varphi_j(x)\}_{j=0}^\infty$ be a smooth partition of unity in R^n subordinate to the covering of open cubes of side 2 with center at these points. We may assume that $\varphi_0(x)\geqq 0$, $\varphi_j(x)=\varphi(x-g_j)$. We set

$$a_{jk}(x, y)=\varphi_j(x)\varphi_k(y)a(x, y).$$

Then we have

$$A=\sum_{j,k=0}^{\infty} A_{jk}, \tag{2}$$

$$A_{jk}f(x)=\int_{R^n} a_{jk}(x, y) \exp (i\lambda S(x, y))f(y)dy. \tag{3}$$

The adjoint $A^*_{j'k'}$ of $A_{j'k'}$ is given by

$$A^*_{j'k'}f(y)=\int \overline{a_{j'k'}(z, y)} \exp (-i\lambda S(z, y))f(z)dz. \tag{4}$$

The kernel function $k(x, y)$ of the operator $A_{jk}A^*_{j'k'}$ turns out to be

$$\begin{aligned} k(x, z)=\varphi_j(x)\varphi_{j'}(z)\int_{R^n} & a(x, y)\overline{a(z, y)}\varphi_k(y)\varphi_{k'}(y) \\ & \times \exp i\lambda(S(x, y)-S(z, y))dy. \end{aligned} \tag{5}$$

Let $L=-i\Big(\sum_j^n \phi_j \dfrac{\partial}{\partial y_j}\Big)/\Phi^2$, where $\phi_j=\dfrac{\partial}{\partial y_j}(S(x, y)-S(z, y))$, $j=1, 2, \cdots, n$. Then $(L-\lambda) \exp (i\lambda(S(x, y)-S(z, y))=0$. Hence we have

$$\begin{aligned} k(x, z)=\lambda^{-l}\varphi_j(x)\varphi_{j'}(z) \int_{R^n} & L^{*l}(a(x, y)\overline{a(z, y)}\varphi_k(y)\varphi_{k'}(y)) \\ & \times \exp i\lambda(S(x, y)-S(z, y))dy. \end{aligned} \tag{6}$$

Here l is an arbitrary nonnegative integer. We use (A-II), (A-III) and (A-IV) and have estimate

$$|L^{*l}(a(x, y)\overline{a(z, y)}\varphi_k(y)\varphi_{k'}(y))|\leqq C\theta(|x-z|)^{-l} \tag{7}$$

if $\operatorname{supp} \varphi_k \cap \operatorname{supp} \varphi_{k'} \ni y$. Therefore we obtain

$$|k(x, z)|\leqq C\lambda^{-l}\varphi_j(x)\varphi_{j'}(z)\theta(|x-z|)^{-l}\chi(g_k-g_{k'}), \tag{8}$$

where χ is the characteristic function of the set $\{x; x<10\sqrt{n}\}$. Let ρ be an arbitrary positive number $\rho<1$. We divide the integral

$$\int_{R^n}|k(x, z)|dz=\int_{|z-x|<\rho}|k(x, z)|dz+\int_{\rho<|z-x|}|k(x, z)|dz. \tag{9}$$

First we have

$$\begin{aligned} \int_{|z-x|<\rho}|k(x, z)|dz & \leqq C\chi(g_k-g_{k'})|\varphi_j(x)|\int_{|z-x|<\rho}\varphi_j(z)dz\,\chi(g_j-g_{j'}) \\ & \leqq C\chi(g_k-g_{k'})\chi(g_j-g_{j'})\rho^n. \end{aligned} \tag{10}$$

If $g_j-g_{j'} \notin \operatorname{supp}\chi$, then $|z-x|>4\sqrt{n}>\rho$ for any (x, z) in support of $\varphi_j(x)\varphi_{j'}(z)$. From this we have

$$\begin{aligned} \int_{\rho<|x-z|}|k(x, z)|dz & \leqq C\lambda^{-l}\chi(g_k-g_{k'})\varphi_j(x)\int_{4\sqrt{n}<|x-z|}\theta(x-z)^{-l}\varphi_{j'}(z)dz \\ & \leqq C\lambda^{-l}\chi(g_k-g_{k'})\theta(|g_j-g_{j'}|)^{-l}, \qquad l=0, 1, \cdots. \end{aligned} \tag{11}$$

On the other hand, if $g_j-g_{j'} \in \operatorname{supp}\chi$, we have

$$\begin{aligned} \int_{\rho<|x-z|}|k(x, z)|dz & \leqq C\lambda^{-l}\chi(g_k-g_{k'})\chi(g_j-g_{j'})\int_{\rho<|x-z|<10\sqrt{n}}|x-z|^{-l}dz \\ & \leqq C\lambda^{-l}\chi(g_k-g_{k'})\chi(g_j-g_{j'})(1+\rho^{n-l}). \end{aligned} \tag{12}$$

Hence we obtain

$$\int |k(x,z)|dz \leqq C\chi(g_k-g_{k'})\chi(g_j-g_{j'})(\rho^n+\lambda^{-l}(1+\rho^{n-l})) + C\lambda^{-l}\chi(g_k-g_{k'})(1+\theta(|g_j-g_{j'}|))^{-l}, \tag{13}$$

for any $\rho\in[0,1]$ and $l=0,1,2,\cdots$. We choose $\rho=\lambda^{-1}$ and $l>2\max(n,n/\sigma)$. Then we have

$$\int_{R^n} |k(x,z)|dz \leqq C\lambda^{-n}\chi(g_k-g_{k'})(1+\theta(|g_j-g_{j'}|))^{-l}. \tag{14}$$

Similarly we have

$$\int_{R^n} |k(x,z)|dx \leqq C\lambda^{-n}\chi(g_k-g_{k'})(1+\theta(|g_j-g_{j'}|))^{-l}. \tag{15}$$

It follows from (14) and (15) that

$$\|A_{jk}A^*_{j'k'}\| \leqq C\lambda^{-n}\chi(g_k-g_{k'})(1+\theta(|g_j-g_{j'}|))^{-l}. \tag{16}$$

Note that the kernel function $k_1(x,z)$ of transformation $A^*_{jk}A_{j'k'}$ is

$$k_1(x,z)=\varphi_k(x)\varphi_{k'}(z)\int_{R^n}\overline{a(y,x)}a(y,z)\exp -i\lambda(S(y,x)-S(y,z))dy. \tag{17}$$

The same discussion as above proves that

$$\|A^*_{jk}A_{j'k'}\| \leqq C\lambda^{-n}\chi(g_j-g_{j'})(1+\theta(|g_k-g_{k'}|))^{-l}. \tag{18}$$

We set $p=(j,k)$ and $p'=(j',k')$ in Z^{2n}. Then we have

$$\|A^*_pA_{p'}\| \leqq h^2(p,p') \tag{19}$$

and

$$\|A_pA^*_{p'}\| \leqq h^2(p,p'), \tag{20}$$

where

$$h(p,p')=C\lambda^{-n/2}(\chi(g_j-g_{j'})(1+\theta(|g_k-g_{k'}|))^{-l} + \chi(g_k-g_{k'})(1+\theta(|g_j-g_{j'}|))^{-l})^{1/2}.$$

We can easily see that $\sup_{p'}(\sum_p h(p,p')\leqq C\lambda^{-1/2n}$. This and lemma of Calderòn-Vaillancourt prove our theorem.

§ 5. **A corollary.** The above result is applicable to integral transformation of the following type:

$$Bf(x)=\iint_{R^n\times R^n} a(x,y)\exp i\lambda(S(x,y)-y\cdot z)f(z)dzdy. \tag{21}$$

Corollary. *Assume that functions $S(x,y)$ and $a(x,y)$ satisfy assumptions* (A-I), (A-II), (A-III) *and* (A-IV). *Then the integral transformation B defined by* (21) *is estimated as*

$$\|Bf\| \leqq C\lambda^{-n}\|f\|,$$

where $C>0$ is a constant independent of f and λ.

Proof. Set

$$g_\lambda(y)=\int_{R^n}\exp(-i\lambda y\cdot z)f(z)dz.$$

Then we have $\|g_\lambda\|=(\lambda/2\pi)^{-n/2}\|f\|$. We apply our theorem to

$$Bf(x)=\int_{R^n} a(x,y)\exp i\lambda S(x,y)g_\lambda(y)dy.$$

We obtain $\|Bf\| \leqq C\lambda^{-n/2}\|g_\lambda\| \leqq C\lambda^{-n}\|f\|$.

References

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