

19. Theory of Tempered Ultrahyperfunctions. I

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In this paper we consider the tempered ultrahyperfunction which was introduced by Sebastião e Silva [3] and Hasumi [1] in the name of tempered ultradistribution. We will give some precisions on the work of M. Hasumi. The same idea was developed in [2] for the Fourier ultrahyperfunction.

§ 1. The basic spaces $H(\mathbf{R}^n; O')$ and $H(\mathbf{R}^n; K')$. Let $K' \subset \mathbf{R}^n$ be a convex compact set. Put $h_{K'}(x) = \sup \{ \langle x, \xi \rangle; \xi \in K' \}$, $\langle x, \xi \rangle$ being the canonical inner product of $\mathbf{R}^n \times \mathbf{R}^n$. Remark $h_{K'}(x) = k'|x|$, $|x| = |x_1| + |x_2| + \cdots + |x_n|$ for $K' = [-k', k']^n$. Let $H_b(\mathbf{R}^n; K')$ be the space of all C^∞ functions f on \mathbf{R}^n such that $\exp(h_{K'}(x))D^p f(x)$ is bounded in \mathbf{R}^n for any multi-index p . D^p denotes the partial differential operator

$$\frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}, \quad p = (p_1, p_2, \cdots, p_n), \quad |p| = p_1 + p_2 + \cdots + p_n.$$

We define in $H_b(\mathbf{R}^n; K')$ seminorms

$$(1) \quad \|f\|_{K', k} = \sup \{ \exp(h_{K'}(x)) |D^p f(x)|; |p| \leq k, x \in \mathbf{R}^n \}$$

for $k = 0, 1, 2, \cdots$. With these seminorms, the space $H_b(\mathbf{R}^n; K')$ is a Fréchet space. If K'_1 and K'_2 are two convex compact sets in \mathbf{R}^n such that $K'_1 \subset K'_2$, then the canonical injection

$$(2) \quad H_b(\mathbf{R}^n; K'_2) \hookrightarrow H_b(\mathbf{R}^n; K'_1)$$

is continuous.

Let O' be a convex open set of \mathbf{R}^n . We define

$$(3) \quad H(\mathbf{R}^n; O') = \lim_{K' \subset O'} \text{proj} H_b(\mathbf{R}^n; K'),$$

where K' runs through the convex compact sets contained in O' and the projective limit is taken following the canonical injections (2). If O'_1 and O'_2 are two convex open sets in \mathbf{R}^n such that $O'_1 \subset O'_2$, we have the canonical injection: $H(\mathbf{R}^n; O'_2) \hookrightarrow H(\mathbf{R}^n; O'_1)$.

For a convex compact set K' of \mathbf{R}^n , we put

$$(4) \quad H(\mathbf{R}^n; K') = \lim_{K'' \supset K'} \text{ind} H_b(\mathbf{R}^n; K''),$$

where K'' runs through the convex compact sets such that K' is contained in the interior of K'' and the inductive limit is taken following the canonical mappings (2). If K'_1 and K'_2 are convex compact sets in \mathbf{R}^n such that $K'_1 \subset K'_2$, then we have the canonical injection: $H(\mathbf{R}^n; K'_2) \hookrightarrow H(\mathbf{R}^n; K'_1)$.

Theorem 1. *Let O' be a convex open set in \mathbf{R}^n and K' be a convex*

compact set in \mathbf{R}^n . Then the space $H(\mathbf{R}^n; O')$ is an FS space and the space $H(\mathbf{R}^n; K')$ is the inductive limit of FS spaces.

Proof. Choose a sequence of convex compact sets K'_m such that $K'_m \subset K'_{m+1}$ and $O' = \bigcup_{m=1}^{\infty} K'_m$. For any integer $m > 0$ and $f \in C^m(\mathbf{R}^n)$, put $\|f\|_m = \|f\|_{K'_m, m}$. Then clearly $\|f\|_{m+1} \geq \|f\|_m$. Put

$$X_m = \{f \in C^m(\mathbf{R}^n); \|f\|_m < \infty\}.$$

Then we can show easily that X_m is a Banach space and that the mapping $X_{m+1} \rightarrow X_m$ is compact. As we have $H(\mathbf{R}^n; O') = \lim \text{proj } X_m$, the space $H(\mathbf{R}^n; O')$ is an FS space. The second statement results from the equality:

$$H(\mathbf{R}^n; K') = \lim_{O' \supset K'} \text{ind } H(\mathbf{R}^n; O'),$$

where O' runs through the convex open sets containing K' and the inductive limit is taken following the canonical injections. q.e.d.

Theorem 2. Let $\mathcal{D}(\mathbf{R}^n)$ be the space of all C^∞ functions on \mathbf{R}^n with compact support. Then $\mathcal{D}(\mathbf{R}^n)$ is dense in $H(\mathbf{R}^n; O')$ and in $H(\mathbf{R}^n; K')$.

Proof. Fix a function $\alpha(x) \in \mathcal{D}(\mathbf{R}^n)$ such that $\alpha(x) \equiv 1$ for $|x| < 1$, $\alpha(x) \equiv 0$ for $|x| > 2$ and $0 \leq \alpha(x) \leq 1$. If $f \in H(\mathbf{R}^n; O')$ (resp. $H(\mathbf{R}^n; K')$), then $f_m(x) = \alpha(x/m)f(x) \in \mathcal{D}(\mathbf{R}^n)$ converges to f in the topology of $H(\mathbf{R}^n; O')$ (resp. of $H(\mathbf{R}^n; K')$) as m tends to ∞ . q.e.d.

Corollary. The space $H = H(\mathbf{R}^n; \mathbf{R}^n)$ is dense in $H(\mathbf{R}^n; O')$ and in $H(\mathbf{R}^n; K')$.

§ 2. Spaces of distributions of exponential growth. For a linear topological space (over the field C) X , we denote by X' the dual space of X . Thanks to Theorem 2 and Corollary, we have two inclusion relations:

$$\begin{aligned} H'(\mathbf{R}^n; O') &\subset H' \subset \mathcal{D}'(\mathbf{R}^n), \\ H'(\mathbf{R}^n; K') &\subset H' \subset \mathcal{D}'(\mathbf{R}^n). \end{aligned}$$

It means that a continuous linear functional on the space $H(\mathbf{R}^n; O')$ or on $H(\mathbf{R}^n; K')$ is considered as a Schwartz distribution on \mathbf{R}^n . Hasumi [1] studied the space $H = H(\mathbf{R}^n; \mathbf{R}^n)$ and its dual space H' . He denotes H' by A_∞ and determined it as the space of distributions of exponential growth on \mathbf{R}^n . (See also Zieleźny [4].) The next theorem generalizes Hasumi's results:

Theorem 3. A distribution $T \in \mathcal{D}'(\mathbf{R}^n)$ belongs to $H'(\mathbf{R}^n; O')$ if and only if there exist a multi-index p , a convex compact set $K' \subset O'$ and a bounded continuous function F such that

$$(5) \quad T = D^p[\exp(h_{K'}(x))F(x)].$$

A distribution T belongs to $H'(\mathbf{R}^n; K')$ if and only if for any $\varepsilon > 0$ there exist a multi-index p_ε and a bounded continuous function F_ε such that

$$(5') \quad T = D^{p_\varepsilon}[\exp(h_{K'}(x) + \varepsilon|x|)F_\varepsilon(x)].$$

§ 3. The Fourier image of $H(\mathbb{R}^n; O')$ and $H(\mathbb{R}^n; K')$. For $f \in H(\mathbb{R}^n; O')$ we put

$$(6) \quad \mathcal{F}f(\zeta) = (2\pi)^{-n/2} \int \cdots \int_{\mathbb{R}^n} f(x) \exp(-i\langle x, \zeta \rangle) dx_1 dx_2 \cdots dx_n,$$

where $\langle x, \zeta \rangle = x_1 \zeta_1 + x_2 \zeta_2 + \cdots + x_n \zeta_n$, $\zeta \in \mathbb{C}^n$. The Fourier transformation $\mathcal{F}f$ of f is a function defined for ζ for which the right hand side of (6) has a meaning. To describe the Fourier images, we introduce some spaces of holomorphic functions.

We will use the notation $T(A) = \mathbb{R}^n \times iA$ for a subset A of \mathbb{R}^n . For a convex compact set K' of \mathbb{R}^n , $\mathfrak{S}_b(T(K'))$ is, by definition, the space of all continuous functions φ on $T(K')$ such that φ is holomorphic in the interior $T(\overset{\circ}{K}')$ of $T(K')$ and that $\zeta^p \varphi(\zeta)$ is bounded in $T(K')$ for any multi-index p , where $\zeta^p = \zeta_1^{p_1} \zeta_2^{p_2} \cdots \zeta_n^{p_n}$ for $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ and $p = (p_1, p_2, \dots, p_n)$. We put

$$(7) \quad \|\varphi\|^{K';k} = \sup \{ |\zeta^p \varphi(\zeta)|; \zeta \in T(K'), |p| \leq k \} \quad \text{for } k=0, 1, 2, \dots$$

Endowed with these seminorms, the space $\mathfrak{S}_b(T(K'))$ is a Fréchet space. If $K'_1 \subset K'_2$ are two convex compact sets, we have the canonical injection

$$(8) \quad \mathfrak{S}_b(T(K'_2)) \hookrightarrow \mathfrak{S}_b(T(K'_1)).$$

Let O' be a convex open set in \mathbb{R}^n . Put

$$(9) \quad \mathfrak{S}(T(O')) = \lim_{K' \subset O'} \text{proj } \mathfrak{S}_b(T(K')),$$

where K' runs through the convex compact sets contained in O' and the projective limit is taken following the restriction mappings (8).

Hasumi [1] considered the space $\mathfrak{S} = \mathfrak{S}(T(\mathbb{R}^n))$ and proved the Fourier transformation establishes a topological isomorphism of H onto \mathfrak{S} . As the first step of generalization of Hasumi's results, we can show easily the following proposition.

Proposition 1. *If $f \in H(\mathbb{R}^n; O')$, the Fourier transformation $\mathcal{F}f$ of f belongs to the space $\mathfrak{S}(T(O'))$.*

Let $\varphi \in \mathfrak{S}(T(O'))$. Then, by the Cauchy integral theorem, the integrals

$$\begin{aligned} & \int \cdots \int_{\mathbb{R}^n} \varphi(\xi + i\eta) \exp(-i\langle x, \xi + i\eta \rangle) d\xi_1 d\xi_2 \cdots d\xi_n \\ &= \int \cdots \int_{\mathbb{R}^n + i\eta} \varphi(\zeta) \exp(-i\langle x, \zeta \rangle) d\zeta_1 d\zeta_2 \cdots d\zeta_n \end{aligned}$$

are independent of $\eta \in O'$. Hence we may put for $\eta \in O'$

$$(10) \quad \mathcal{F}\varphi(x) = (2\pi)^{-n/2} \int \cdots \int_{\mathbb{R}^n} \varphi(\xi + i\eta) \exp(-i\langle x, \xi + i\eta \rangle) d\xi_1 d\xi_2 \cdots d\xi_n$$

and call it the Fourier transformation of $\varphi \in \mathfrak{S}(T(O'))$.

Proposition 2. *The Fourier transformation \mathcal{F} defined by (10) maps $\mathfrak{S}(T(O'))$ into $H(\mathbb{R}^n; -O')$. Moreover we have*

$$(11) \quad \mathcal{F}(\mathcal{F}f)(x) = f(-x) \quad \text{for } f \in H(\mathbb{R}^n; O'),$$

$$(11') \quad \mathcal{F}(\mathcal{F}\varphi)(\zeta) = \varphi(-\zeta) \quad \text{for } \varphi \in \mathfrak{S}(T(O')).$$

From Propositions 1 and 2 we can conclude the following theorem:

Theorem 4. *The Fourier transformation (6) establishes a topological isomorphism of $H(\mathbf{R}^n; O')$ onto $\mathfrak{S}(T(O'))$ and the inverse mapping is given by*

$$(10) \quad \overline{\mathcal{F}}\varphi(x) = (2\pi)^{-n/2} \int \cdots \int_{\mathbf{R}^n} \varphi(\xi + i\eta) \exp(i\langle x, \xi + i\eta \rangle) d\xi_1 d\xi_2 \cdots d\xi_n,$$

where $\eta \in O'$.

Let K' be a convex compact set in \mathbf{R}^n . We define

$$(12) \quad \mathfrak{S}(T(K')) = \lim_{K'' \supset K'} \text{ind } \mathfrak{S}_\delta(T(K'')),$$

where K'' runs through the convex compact sets such that K' is contained in the interior of K'' and the inductive limit is taken following the restriction mappings (8).

For $\varepsilon > 0$ we will denote by K'_ε the ε -neighborhood of K' , namely, $K'_\varepsilon = K' + \{\xi \in \mathbf{R}^n; |\xi| < \varepsilon\}$. K'_ε is a convex open set such that $K'_\varepsilon \supset K'$. As we have

$$(13) \quad H(\mathbf{R}^n; K') = \lim_{\varepsilon > 0} \text{ind } H(\mathbf{R}^n; K'_\varepsilon)$$

and

$$(14) \quad \mathfrak{S}(T(K')) = \lim_{\varepsilon > 0} \text{ind } \mathfrak{S}(T(K'_\varepsilon)),$$

we can conclude the following theorem from Theorem 4.

Theorem 4'. *The Fourier transformation \mathcal{F} defined by (6) establishes a topological isomorphism of $H(\mathbf{R}^n; K')$ onto $\mathfrak{S}(T(K'))$. The inverse mapping $\overline{\mathcal{F}}$ is given by (10').*

Corollary. *The space $\mathfrak{S}(T(O'))$ is an FS space and $\mathfrak{S}(T(K'))$ is the inductive limit of FS spaces.*

Via the Fourier transformation, Corollary to Theorem 2 gives the following Runge type theorem.

Theorem 5. *The space $\mathfrak{S} = \mathfrak{S}(T(\mathbf{R}^n))$ is dense in $\mathfrak{S}(T(O'))$ and in $\mathfrak{S}(T(K'))$.*

§ 4. The dual Fourier transformation. For $T \in H'(\mathbf{R}^n; O')$ (resp. $H'(\mathbf{R}^n; K')$), we define the dual Fourier transformation $\mathcal{F}_a T$ as a continuous linear functional on $\mathfrak{S}(T(-O'))$ (resp. $\mathfrak{S}(T(-K'))$) by the formula

$$(15) \quad (\mathcal{F}_a T, \varphi) = (T, \mathcal{F}\varphi) \quad \text{for } \varphi \in \mathfrak{S}(T(-O')) \text{ (resp. } \mathfrak{S}(T(-K'))).$$

As a consequence of Theorems 4 and 4', we have the following theorem.

Theorem 6. *The dual Fourier transformation (15) gives topological isomorphisms*

$$(16) \quad \mathcal{F}_a: H'(\mathbf{R}^n; O') \rightarrow \mathfrak{S}'(T(-O'))$$

and

$$(16') \quad \mathcal{F}_a: H'(\mathbf{R}^n; K') \rightarrow \mathfrak{S}'(T(-K')).$$

We will define, in the forthcoming papers, the Fourier transformation of $T \in H'(\mathbf{R}^n; O')$ and $T \in H'(\mathbf{R}^n; K')$ without use of the duality. It will turn out to be identical with the dual Fourier transformation.

References

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