51. *Boundary Behavior of Harmonic Measures*

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Introduction. In the recent study of harmonic functions on an open Riemann surface, it is known that every canonical potential, especially every harmonic measure assumes a constant value quasi-everywhere (or except for harmonic measure zero) on each component of Kuramochi boundary. Such a property for boundary behaviors has been investigated by Kusunoki and some others.

Continuously they have investigated whether these boundary behaviors would characterize those functions. For Riemann surfaces with a finite number of boundary components, these characterizations of harmonic measures are trivially established. M. Watanabe shows that for Riemann surfaces with countably many boundary components, it is also true under some additional conditions, and that there exist Riemann surfaces with uncountably many boundary components in which these boundary behaviors do not always characterize those functions.

In the present paper, we shall show first simple examples showing that such characterizations can not be expected for Riemann surfaces with countably many boundary components. We should note that there are some differences between the characterization by using harmonic measure and that by using Kuramochi capacity. Our Example 1 is concerned with the former and is a planar region. On the other hand, the Riemann surface in Example 2 concerned with the latter is of infinite genus. And next we shall show that the characterization of harmonic measures by using Kuramochi capacity is established for Riemann surfaces with finite genus and countably many boundary components.

At the end, I wish to express my hearty thanks to Professor Tadao Kubo for his kind guidance and encouragement.

1. Let R be a Riemann surface and $HC=HC(R)$ be the set of harmonic functions on R such that i) each u belongs to class HD, i.e. u is harmonic and has a finite Dirichlet integral, ii) u takes a constant value almost everywhere (i.e. except a set of harmonic measure zero) on each Kerékjártó-Stoïlow's boundary component of R. We shall use the same terminologies and notations as in Ahlfors-Sario [1]. We write

$$HM=\{u\in HD\,;\, du\in \Gamma_{hm}\}.$$

We call the elements of HM harmonic measures. Note that we use the same terminology (harmonic measures) for the representing measures of bounded harmonic functions. The harmonic measure of a set X in some boundary is denoted by 1_X, and 1_X is the lower envelope for all equilibrium potentials of neighbourhoods of X. If $1_X \equiv 0$, we say that X is harmonic measure zero. It is known that $HM \subset HC$ (cf. [4]). "A problem is whether HC coincides with HM or not." For this purpose we use the well-known orthogonal decomposition:

$$\Gamma_{he} = \Gamma_{hm} + {}^*\Gamma_{hse} \cap \Gamma_{he}.$$

We write

$$HK = \{u \in HD\,;\, du \in {}^*\Gamma_{hse},\ u(a)=0 \text{ at a fixed point } a \in R\},$$

and

$$KC = HK \cap HC.$$

Our problem is equivalent to ask whether KC consists of only zero function or not. Now a Riemann surface R_c is assumed that it has at most countably many boundary components. At first we observe the class KC.

Lemma. *Let R_c be of finite genus. If $u \in KC$ has a constant value precisely on each Kerékjártó-Stoïlow's boundary component of R_c, then u is a constant zero function.*

Proof. We suppose that there exists such a nonconstant u. When R_c is a planar region, $f = u + i^*u$ is a single valued analytic Dirichlet finite function. The image of R_c by f is a covering surface over complex plane whose countably many boundaries are parallel to imaginary axis, so it has infinite area. This is a contradiction. When R_c is a Riemann surface with finite genus, we can cut R_c along compact curves so that the resulting surface R_c' becomes a planar region. Then $f = u + i^*u$ is a single valued analytic Dirichlet finite function in R_c'. By maximum principle u does not take the maximal value of u on the relative boundary of R_c' in R_c. So we conclude that the image of R_c' by f has infinite area. Similarly this leads to a contradiction.

This lemma suggests the following for general Riemann surfaces.

Proposition 1. *Any function $u \in KC$ has not any compact level cycle $\{x \in R\,;\, u(x) = c\}$.*

Proof. We suppose that u has a compact level cycle $\{x \in R\,;\, u(x) = c\}$. Let V_c be a component of $\{x \in R\,;\, u(x) > c\}$, and cycle γ_c be the relative boundary of V_c which consists in $\{x \in R\,;\, u(x) = c\}$. Since the inner normal derivative of u with respect to V_c is positive on γ_c except for a finite number of points, so $\int_{-\gamma_c} {}^*du > 0$. On the other hand, since γ_c is a dividing cycle and ${}^*du \in \Gamma_{hse}$, $\int_{-\gamma_c} {}^*du = 0$. This is a contradiction, which proves the statement.

Corollary 1. *All values of $u \in KC$ in R are taken at the ideal boundary.*

The statement of Lemma is established without the restriction of genus for R_c. That is to say,

Corollary 2. *If $u \in KC$ has a constant value precisely on each Kerékjártó-Stoïlow's boundary component of R_c, then u is a constant zero function.*

2. Now we shall show *a planar region with countably many boundary components such that* $KC \neq \{0\}$, i.e. $HC \neq HM$.

Example 1. We consider a rectangle

$$R' = \{z = x + iy \in \boldsymbol{C};\ -1 < x < 1,\ -1 < y < 1\},$$

and the following families of slits;

$$E_p = \{z = x + iy \in \boldsymbol{C};\ x = 1 - 1/2^p,\ |y| \leq 1 - 1/p + 1/p^3\},$$
$$E'_p = \{z = x + iy \in \boldsymbol{C};\ -z \in E_p\},$$
$$F_{p,j} = \{z = x + iy \in \boldsymbol{C};\ x = j/p^4,\ |y - (1 - 1/p)| \leq 1/p^3\},$$
$$F'_{p,j} = \{z = x + iy \in \boldsymbol{C};\ \bar{z} \in F_{p,j}\},$$

where p denotes a prime integer and $|j| \leq p^4 - 1$. We set

$$\mathbf{E}_p = \mathbf{E}_p \cup E'_p, \qquad \mathbf{F}_{p,j} = F_{p,j} \cup F'_{p,j},$$

and

$$R = R' - \bigcup_p \left[\mathbf{E}_p \cup \left(\bigcup_j \mathbf{F}_{p,j}\right)\right].$$

Then R is a region with countably many boundary components. Let $\varDelta_\beta$ be an outer boundary of R,

$$\varDelta_\beta = \{z = x + iy \in \boldsymbol{C};\ |x| = 1,\ |y| \leq 1, \text{ or } |y| = 1,\ |x| \leq 1\}.$$

Now we shall show that the harmonic measure of $\varDelta_\beta$ is zero. Consider the rectangles

$$A(p_n) = \{z = x + iy \in \boldsymbol{C};\ |x| \leq 1 - 1/2^{p_n},\ |y| \leq 1 - 1/p_n\},$$
$$B(p_{n+1}) = \{z = x + iy \in R';\ 1 - 1/2^{p_{n+1}} \leq |x| \leq 1,$$
$$\text{or } 1 - 1/p_{n+1} - 2/p_{n+1}^3 \leq |y| \leq 1\},$$

where $\{p_n\}$ is an increasing sequence of prime integers. Let S_n be a positive superharmonic function in R, such that

$$S_n = \begin{cases} 0 & \text{on all } \mathbf{E}_p \text{ with } \mathbf{E}_p \cap A(p_n) \neq \emptyset \\ 0 & \text{on all } \mathbf{F}_{p,j} \text{ with } \mathbf{F}_{p,j} \cap A(p_n) \neq \emptyset \\ 1 & \text{on } B(p_{n+1}) \\ \multicolumn{2}{l}{\text{harmonic in } R - B(p_{n+1}) \text{ except for the above slits.}} \end{cases}$$

Then as above mention the harmonic measure of $\varDelta_\beta$ is equal to the reduced function $1_{\varDelta_\beta}(z)$, and $1_{\varDelta_\beta}(z) \leq S_n(z)$. Here we consider a harmonic function f_p on a rectangle $G_p = \{z = x + iy \in \boldsymbol{C};\ |x| < 1/2p,\ |y| < 1\}$ whose boundary values are 0 on $|x| = 1/2p$, 1 on $|y| = 1$. We set

$$M(p) = \sup\{f_p(x + iy);\ y = 0\}.$$

Clearly $M(p)$ converges to 0, as $p \to +\infty$. Since $S_n(z)$ is bounded by $M(p_n)$ on the boundary of $A(p_n)$ and so on $A(p_n)$. Hence $S_n(z)$ converges to 0 on every compact set for $p_n \to +\infty$. Thus the harmonic measure of Δ_β is zero.

Now evidently the nonconstant function x belongs to HC. Since $dx \in {}^*\Gamma_{hse}$, the class KC contains nonconstant functions.

Remark. When we consider the Kuramochi's compactification of R, Δ_β is *not of Kuramochi capacity* 0. To see this let $\hat{R}$ be the double of R along all slits $\{\mathbf{E}_p, \mathbf{F}_{p,j}\}$ of R. Since we can regard y as a harmonic Dirichlet function on $\hat{R}$, $\hat{R}$ is a hyperbolic Riemann surface. Let R_0 be a disk in R and R'_0 be a disk in $\hat{R}-R$ which is mapped from R_0 by natural anti-conformal mapping of $\hat{R}$. Let $\{\hat{R}_n\}$ be a regular exhaustion of $\hat{R}$ such that $\hat{R}_1$ contains $R_0 \cup R'_0 = \hat{R}_0$. The harmonic function W_n in R_n whose boundary values are 0 on $\partial\hat{R}_0$, 1 on $\partial\hat{R}_n$ converges to a nonconstant positive harmonic function W in $\hat{R}-\bar{\hat{R}}_0$ as $n \to +\infty$. The restriction of W in $R-\bar{R}_0$ is a nonzero capacitary potential of Δ_β in the sense of Kuramochi. Hence Δ_β is not of Kuramochi capacity 0.

3. Next we investigate a function class $HC^*(\subset HD)$ whose condition is stronger than that of HC. Each element of HC^* has a quasi continuous extension to the Kuramochi's compactification and a constant value quasi-everywhere on every boundary component Δ_e corresponding to every Kerékjártó-Stoïlow's boundary component (cf. [2] Theorem 2). We set $KC^* = HK \cap HC^*$.

We shall show that there exists *a Riemann surface with countably many boundary components* $\{\Delta_e\}$ *such that* $KC^* \neq \{0\}$, *i.e.* $HC^* \neq HM$.

Example 2. Let R' and $\mathbf{E}_p$ be the same as in Example 1, and let $\{\mathbf{L}_p\}$ be the set of slits such that

$$\mathbf{L}_p = \{z = x+iy \in \boldsymbol{C};\ |x| \leq 1-1/2^p - d(p),\ |y| = 1-1/p+1/p^3\}.$$

We need to choose a proper real positive number $d(p)$. We consider a rectangle G^{n+1} and the set of slits $\{\mathbf{L}^n_\varepsilon\}_\varepsilon$

$$G^{n+1} = \{z = x+iy \in \boldsymbol{C};\ |x| < 1-1/2^{p_{n+1}},\ |y| < 1-1/p_{n+1}+1/p^3_{n+1}\},$$

$$\mathbf{L}^n_\varepsilon = \{z = x+iy \in \boldsymbol{C};\ 1-1/2^{p_n}-\varepsilon \leq |x| \leq 1-1/2^{p_n},\ |y| = 1-1/p_n+1/p^3_n\}.$$

Let u^n_ε be a harmonic function with minimal Dirichlet integral among the functions in $G^{n+1}-G^n$ whose boundary values are 1 on ∂G^{n+1}, 0 on $\mathbf{L}^n_\varepsilon$. Then by Dirichlet principle, Dirichlet integral $D(u^n_\varepsilon)$ of u^n_ε increases with ε. Moreover we can see

$$\lim_{\varepsilon \to 0} D(u^n_\varepsilon) = 0.$$

So we can choose $d(p_n)$ so that $D(u^n_{d(p_n)}) < 1/p_n$. Denote

$$R_2 = R' - \Big(\bigcup_p \mathbf{E}_p\Big) \cup \Big(\bigcup_p \mathbf{L}_p\Big).$$

Let $\hat{R}_2$ be the double of R_2 along the slits $\{\mathbf{L}_p\}$. Then $\hat{R}_2$ is a Riemann surface with infinite genus and countably many boundary components. Let $\hat{G}^{n+1}_2$ be the union of $G^{n+1} \cap R_2$ and its image by natural anti-conformal mapping in $\hat{R}_2$. $K_n = \hat{R}_2 - \hat{G}^{n+1}_2$ is a closed set. Let R_0 be a disk which is contained in $\hat{G}_2^2$. The capacitary potential $1_{\tilde{K}_n}$ of K_n is a function in R which has minimal Dirichlet integral among functions with values 1 on K_n and 0 in R_0. We denote by Δ_r an ideal outer boundary component excluded $\{\mathbf{E}_p\}$ and its image $\{\tilde{\mathbf{E}}_p\}$ from whole boundary of $\hat{R}_2$. By Dirichlet principle we have the following inequality

$$D_{\hat{R}}(1_{\tilde{\Delta}_r}) \leq D_{\hat{R}}(1_{\tilde{K}_n}) \leq 2D(u^n_{d(p_n)}) < 2/p_n.$$

Therefore it follows that Δ_r is of Kuramochi capacity 0. From the construction of $\hat{R}_2$ x is regarded a harmonic function on $\hat{R}_2$ and it is easily seen that dx belongs to $^*\Gamma_{hse}$. Since x has a constant value quasi-everywhere on each Δ_e, we conclude that $KC^* \neq \{0\}$ i.e. $HC^* \neq HM$.

4. Moreover we can investigate the case when R is a Riemann surface with finite genus and countably many Kerékjártó-Stoïlow's boundary components.

Proposition 2. *When R_c is a Riemann surface with finite genus and countably many Kerékjártó-Stoïlow's boundary components, $KC^* = \{0\}$ and $HC^* = HM$.*

Our problems are connected with generalized normal derivatives which are used to prove the first mentioned property of harmonic measures, and from Proposition 4 in [2] directly we get the following.

Corollary 1. *When R is a Riemann surface with finite genus and countably many Kerékjártó-Stoïlow's boundary components, the set of HK-functions with generalized normal derivatives is dense in the space of HK with respect to Dirichlet norm.*

Remark. In the region R of Example 1 the set of HK-functions with generalized normal derivatives is dense in the space HK, but the set of HK-functions with Γ_{he}-generalized normal derivatives is not dense in the space of HK (cf. [6]), which shows the differences between generalized normal derivatives and Γ_{he}-generalized normal derivatives. In the Riemann surface $\hat{R}_2$ of Example 2, the set of HK-functions with generalized normal derivatives is not dense in the space of HK.

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