86. On the Yukawa-coupled Klein-Gordon-Schrödinger Equations in Three Space Dimensions

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1. Introduction and notation. We consider the Yukawa-coupled Klein-Gordon-Schrödinger equations in R^3:

$$(1)\quad \begin{aligned} &i\frac{\partial\psi(t,x)}{\partial t}+\Delta\psi(t,x)=g\psi(t,x)\phi(t,x),\\ &\left(\Delta-\frac{\partial^2}{\partial t^2}-\mu^2\right)\phi(t,x)=g\psi(t,x)\overline{\psi(t,x)},\end{aligned}$$

which represent the classical model of dynamics of conserved complex nucleon fields ψ interacting with neutral real scalar meson fields ϕ. The constant μ describes mass of a meson and g a coupling real constant.

In the case of one space dimension, the existence of global solutions of the Cauchy problem has been established by the authors [3]. In the case of relativistic fields, that is, when nucleons are governed by the Dirac spinor fields, we must treat the coupled Klein-Gordon-Dirac equations:

$$\left(i\gamma_\nu\frac{\partial}{\partial x_\nu}-m\right)\psi=g\psi\phi \qquad \left(\frac{\partial}{\partial x_0}=\frac{\partial}{\partial t}\right),$$
$$\left(\Delta-\frac{\partial^2}{\partial t^2}-\mu^2\right)\phi=g\psi\bar{\psi},$$

which were investigated by Chadam and Glassey [1], [2].

In this paper, our purpose is to state the existence and uniqueness theorems for global solutions of the initial-boundary value problem for the system (1) in Ω with boundary conditions:

$$(2)\qquad \psi(t,x)=\phi(t,x)=0 \quad \text{for } x\in\partial\Omega \quad \text{and} \quad t\geqslant 0$$

and initial conditions:

$$(3)\quad \psi(0,x)=\psi_0(x),\ \phi(0,x)=\phi_0(x) \quad \text{and} \quad \phi_t(0,x)=\phi_1(x) \quad \text{for } x\in\Omega,$$

where Ω denotes a bounded domain in R^3 with sufficiently smooth boundary $\partial\Omega$.

In section 2, we refer to the global existence theorem of the initial-boundary value problem (1)–(3), and the main tool for proving them. In section 3, we represent the uniqueness result. In section 4, we investigate the regularity properties of solutions of (1)–(3).

In this note, we state the results only. Detailed proofs will be published elsewhere.

Notations. Let $L^2(\Omega)$ be the complex or real space of square integrable functions with inner product and norm:

$$(u, v)=\int_\Omega u(x)\overline{v(x)}dx, \qquad \|u\|=(u, u)^{1/2}.$$

$H^m(\Omega)$ ($m(\geqslant 1)$: integer) denote the complex or real Sobolev spaces equipped with inner product and norm:

$$(u, v)_m=\sum_{j\leqslant m}\int_\Omega D^j u(x)\overline{D^j v(x)}dx, \qquad \|u\|_m=(u, u)_m^{1/2}.$$

Let $C_0^\infty(\Omega)$ denote the set of all infinitely differentiable complex-valued or real-valued functions with compact support in Ω. The spaces $H_0^m(\Omega)$ are the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^m(\Omega)$. Since Ω is bounded, in the space $H_0^1(\Omega)$ $\|\cdot\|_1$ is equivalent to $\|\nabla\cdot\|$. Therefore, $(\nabla\cdot, \nabla\cdot)$ is employed as the inner product of $H_0^1(\Omega)$ and we denote it by $((\cdot,\cdot))$. For a Hilbert space X, $L^\infty(0, T: X)$ denotes the space of measurable functions on $[0, T]$ with values in X and $C^k(0, T: X)$ denotes the space of k-times continuously differentiable functions on $[0, T]$ with values in X.

2. Global existence. Suppose that $[\psi, \phi]$ are smooth solutions of (1)–(3), the following energy identities are valid:

$$\|\psi(t)\|^2=\|\psi_0\|^2, \tag{4}$$

$$\begin{aligned}&\frac{1}{2}\|\nabla\phi(t)\|^2+\frac{1}{2}\|\phi_t(t)\|^2+\frac{\mu^2}{2}\|\phi(t)\|^2\\ &\quad+\|\nabla\psi(t)\|^2+g\int_\Omega|\psi(t, x)|^2\phi(t, x)dx\\ &=\frac{1}{2}\|\nabla\phi_0\|^2+\frac{1}{2}\|\phi_1\|^2+\frac{\mu^2}{2}\|\phi_0\|^2+\|\nabla\psi_0\|^2+g\int_\Omega|\psi_0(x)|^2\phi_0(x)dx.\end{aligned} \tag{5}$$

In virtue of this energy identities, we can establish the global existence theorem of weak solutions of (1)–(3) by using the Galerkin's method and compactness arguments. Moreover, from a priori estimates for higher order derivatives, we can show the existence theorem of strong solutions of (1)–(3). The following theorems are obtained.

Theorem 1 (existence of weak solutions). *Suppose that* $\psi_0\in H_0^1(\Omega)$, $\phi_0\in H_0^1(\Omega)$ *and* $\phi_1\in L^2(\Omega)$. *Then there exists at least a couple of weak solutions* $[\psi, \phi]$ *of* (1)–(3) *satisfying*: $\psi\in L^\infty(0, T: H_0^1(\Omega))$, $\phi\in L^\infty(0, T: H_0^1(\Omega))$ *with* $\phi_t\in L^\infty(0, T: L^2(\Omega))$ *and the integral identities*:

$$\begin{aligned}&\int_0^T[-i(\psi(t), \Psi_t(t))-((\psi(t), \Psi(t)))-g(\psi(t)\phi(t), \Psi(t))]dt\\ &\quad=i(\psi(0), \Psi(0))\end{aligned} \tag{6}$$

for any complex function $\Psi\in C^1(0, T: L^2(\Omega))\cap C^0(0, T: H_0^1(\Omega))$ *such that* $\Psi(T)=0$,

$$\begin{aligned}&\int_0^T[((\phi(t), \Phi(t)))-(\phi_t(t), \Phi_t(t))+\mu^2(\phi(t), \Phi(t))+g(\psi(t)\overline{\psi(t)}, \Phi(t))]dt\\ &\quad=(\phi_t(0), \Phi_t(0))\end{aligned} \tag{7}$$

for any real function $\Phi \in C^1(0, T: L^2(\Omega)) \cap C^0(0, T: H_0^1(\Omega))$ *such that* $\Phi(T) = 0$.

Theorem 2 (existence of strong solutions). *If* $\psi_0 \in H_0(\Omega) \cap H^3(\Omega)$, $\phi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $\phi_1 \in H_0^1(\Omega)$, *then there exists a couple of strong solutions* $[\psi, \phi]$ *of* (1)–(3) *such that*:

$$\psi \in L^\infty(0, T: H_0^1(\Omega) \cap H^3(\Omega)) \quad \text{with} \quad \psi_t \in L^\infty(0, T: L^2(\Omega)),$$
$$\phi \in L^\infty(0, T: H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{with } \phi_t \in L^\infty(0, T: H_0^1(\Omega)) \quad \text{and}$$
$$\phi_{tt} \in L^\infty(0, T: L^2(\Omega)).$$

3. Uniqueness. **Theorem 3** (uniqueness of strong solutions). *A couple of strong solutions* $[\psi, \phi]$ *of the problem* (1)–(3) *is uniquely determined by the initial conditions.*

Remark 1. In the case of one space dimension, the uniqueness theorem of weak solutions in the sense of the class mentioned in Theorem 1 is obtained in [3]. However, in the case of three space dimensions, the problem whether the weak solution is uniquely determined by the initial conditions or not is still open.

4. Regularity. Let $[\psi, \phi]$ be smooth solutions of (1)–(3). Then the following identity is valid:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla D_t^n\phi(t)\|^2 + \|D_t^{n+1}\phi(t)\|^2 + \mu^2\|D_t^n\phi(t)\|^2 + 2\|\nabla D_t^n\psi(t)\|^2 \\
&\quad + 2g\textstyle\sum_{j=0}^{n-1} {}_nC_j \displaystyle\int_\Omega (D_t^j\psi(t,x)D^{n-j}\overline{\psi(t,x)}D_t^n\phi(t,x))dx \\
&\quad + 2g\textstyle\sum_{j=0}^{n-1} {}_nC_j \displaystyle\int_\Omega (D_t^n\psi(t,x)D_t^j\overline{\psi(t,x)}D_t^{n-j}\phi(t,x))dx \\
&\quad + 2g\textstyle\sum_{j=0}^{n-1} {}_nC_j \displaystyle\int_\Omega (D_t^{n-j}\psi(t,x)D_t^n\overline{\psi(t,x)}D_t^j\phi(t,x))dx) \\
&= g\textstyle\sum_{j=1}^{n-1} {}_{n+1}C_j \displaystyle\int_\Omega (D_t^{n+1-j}\psi(t,x)D_t^j\overline{\psi(t,x)}D_t^j\phi(t,x))dx \\
&\quad + g\textstyle\sum_{j=1}^{n-1} {}_{n+1}C_j \displaystyle\int_\Omega (D_t^j\psi(t,x)D_t^{n+1-j}\overline{\psi(t,x)}D_t^n\phi(t,x))dx \\
&\quad + g\textstyle\sum_{j=1}^{n-1} {}_{n+1}C_j \displaystyle\int_\Omega (D_t^n\psi(t,x)D_t^j\overline{\psi(t,x)}D_t^{n+1-j}\phi(t,x))dx \\
&\quad + g\,{}_nC_{n-1}\int_\Omega D_t^n\psi(t,x)D_t\overline{\psi(t,x)}D_t^n\phi(t,x)dx \\
&\quad + g\,{}_nC_{n-1}\int_\Omega D_t^n\psi(t,x)D_t^n\overline{\psi(t,x)}D_t\phi(t,x)dx \\
&\quad + g\,{}_nC_{n-1}\int_\Omega D_t\psi(t,x)D_t^n\overline{\psi(t,x)}D_t^n\phi(t,x)dx.
\end{aligned}
\tag{8}
$$

From this identity, we obtain inductively desired a priori estimates for higher order derivatives of solutions after a long calculation. Thus, using the projection method of the Galerkin's type with a special system of bases, we have:

Theorem 4. *Suppose* $\psi_0 \in H_0^1(\Omega) \cap H^{2n-1}(\Omega)$, $\phi_0 \in H_0^1(\Omega) \cap H^{2n-3}(\Omega)$ *and* $\phi_1 \in H_0^1(\Omega) \cap H^{\max(2n-5,3)}(\Omega)$ ($n(\geqslant 3)$: *integer*). *Then, the solutions*

$[\psi, \phi]$ *of* (1)–(3) *have the following properties*:

$$D_t^j \psi \in L^\infty(0, T: H_0^1(\Omega) \cap H^{2n-1-2j}(\Omega)) \qquad (j=0,1,2,\cdots,n-1.),$$
$$D_t^j \phi \in L^\infty(0, T: H_0^1(\Omega) \cap H^{n-j}(\Omega)) \qquad (j=0,1,2,\cdots,n-1.),$$
$$D_t^n \phi \in L^\infty(0, T: L^2(\Omega)).$$

Corollary. *If* ψ_0, ϕ_0 *and* ϕ_1 *belong to* $C^\infty(\bar{\Omega}) \cap H_0^1(\Omega)$, *then the solutions* $[\psi, \phi]$ *of* (1)–(3) *are infinitely differentiable on* $[0, T] \times \bar{\Omega}$.

Remark 2. In the case of the Cauchy problem for the coupled Klein-Gordon-Schrödinger equations in three space dimensions, it seems difficult to obtain the same result as Theorem 4 by the Galerkin's method. However, the desired results are obtained if we employ the other method.

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