81. *A Note on Isolated Singularity. II*

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0. Introduction. This is a brief résumé of the second half of the study whose first part has already been announced [3]. The main purpose is to investigate the structure of an isolated singularity when it admits a C^*-action, especially, to obtain some formula concerning the characters of the representations of C^* over various cohomology groups associated with the singularity.

1. Basic concepts. A C^*-action over an isolated singularity (X, x) is a family $T(c), c \in C^*$ of analytic homeomorphisms of X onto itself satisfying that $T(c)x = x$, $T(cc') = T(c)T(c')(c, c' \in C^*)$, and that $T: X \times C \ni (x, c) \to T(z)c \in X$ is analytic. Throughout this note we assume that the constants are the only invariant elements of $\Omega^0_{X,x}$ under this action. Let ξ be the generating vector field of this action. The interior multiplication $i(\xi)$ is an anti-derivation of $\Omega^{\cdot}_X$ regarded as the sheaf of graded algebra. It is well known that the Poincaré complex $\Omega^{\cdot}_X$ is acyclic in this case. However we have some more

Lemma 1. *Under the above condition the sequences*

$$\cdots \xrightarrow{d} \mathcal{H}^0_x(\Omega^p_X) \xrightarrow{d} \mathcal{H}^0_x(\Omega^{p+1}_X) \xrightarrow{d} \cdots$$

$$\cdots \longrightarrow \Omega^p_X \xrightarrow{i(\xi)} \Omega^{p-1}_X \xrightarrow{i(\xi)} \cdots \xrightarrow{i(\xi)} \Omega^0_X \xrightarrow{\alpha} (\iota_x)_* C \longrightarrow 0$$

are exact, where ι_x denotes the inclusion $x \hookrightarrow X$ and α the average map $\Omega^0_{X,x} \ni f \to \int_0^1 T(e^{2\pi i\theta})^* f d\theta \in (\iota_x)_* C_x$.

If we set $\Omega^p_\xi = i(\xi)\Omega^{p+1}_X$, then we have the short exact sequences $0 \to \Omega^p_\xi \to \Omega^p_X \to \Omega^{p-1}_\xi \to 0$. From these we obtain the following Gysin type sequences

$$0 \longrightarrow \mathcal{H}^0_x(\Omega^p_\xi) \longrightarrow \mathcal{H}^0_x(\Omega^p_X) \longrightarrow \mathcal{H}^0_x(\Omega^{p-1}_\xi) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{H}^q_x(\Omega^p_\xi) \longrightarrow \mathcal{H}^q_x(\Omega^p_x) \longrightarrow \mathcal{H}^q_x(\Omega^{p-1}_\xi) \longrightarrow \cdots.$$

Using these, we can prove

Theorem 1. *Let the notation and the assumption be as above. Assume that (X, x) satisfies the condition* (L). *Then $\mathcal{H}^q_x(\Omega^p_\xi) = 0$ for (p, q) such that $p + q \neq \dim X$, $q \neq p+1$, $q < \dim X$, and there are natural isomorphisms $\mathcal{H}^q_x(\Omega^p_X) \simeq \mathcal{H}^q_x(\Omega^{p+1}_X)$ for (p, q) such that $p + q = \dim X$, $0 < q < \dim X$.*

Remark. If $\dim X$ is even, the proof requires some technique from Kähler geometry, though we can avoid the use of this in case dim X is odd.

2. Formula for characteristic function. In the rest of this note we assume that (X, x) satisfies the condition (L). Let $0 \leqq q < n = \dim X$ and denote by $\chi_X^q(t)$ the character of the representation of C^* on $\mathcal{H}_x^q(\Omega_X^{n-q})$ (or on $\mathcal{H}_x^q(\Omega^{n-q+1})$ in view of Theorem 1 if $q>0$); that is,

$$\chi_X^q(t) = \mathrm{Tr}\,(T(t)^* \mid \mathcal{H}_x^q(\Omega_X^{n-q}))$$

where the notation in the largest parenthesis on the right denotes the endomorphism of $\mathcal{H}_x^q(\Omega_X^{n-q})$ induced by the action $T(t)$. These are rational functions in t. In view of the Serre type duality we have $\chi_X^q(t) = \chi_X^{n-q+1}(t^{-1})$ for $2 \leqq q \leqq n-1$, so it is convenient to set $\chi_X^n(t) = \chi_X^1(t^{-1})$, $\chi_X^{n+1}(t) = \chi_X^0(t^{-1})$; further we define the characteristic function of the C^*-action by

$$\chi_X(s, t) = \sum_{q=0}^{n+1} \chi_X^q(t) s^q.$$

Now let f be an analytic function on X such that $df_z \neq 0$ for $z \in X \backslash x$, $T(c)^* f = c^d f$ $(d>0)$. Then $f(x)=0$ and $(Y, y) = (f^{-1}(0), x)$ is a new isolated singularity over which the action $T(c), c \in C^*$ induces a C^*-action. We consider $\chi_Y^0(t), \chi_Y^1(t), \cdots, \chi_Y^n(t), \chi_Y(s, t)$ to be defined similarly. Then, using some argument in proving the result of [3], we obtain

Theorem 2. *Let the assumption and the notation be as above. Then*

$$(1)\qquad \begin{aligned} & s(\chi_Y(s,t) - s^n \chi_Y^0(t^{-1})) - t^d(\chi_Y(s,t) - \chi_Y^0(t)) \\ &= (t^d - 1)(\chi_X(s,t) - \chi_X^0(t) - s^{n+1}\chi_X^0(t^{-1})). \end{aligned}$$

Remark. The characters $\chi_X^0(t), \chi_Y^0(t)$ are in a sense computable; for example, according to Lemma 1, $\chi_X^0(t)$ is equal to an alternating sum of the characters on the spaces $\Omega_{X,x}^p$, $p>n$; to determine these spaces from the defining equation of (X, x) is easier comparing with the determination of the cohomology groups $\mathcal{H}_x^q(\Omega_X^p)$.

3. Application. Let $H_X^{p,q}$ $(0 \leqq q < n-1)$, $H_Y^{p,q}$ $(0 \leqq q < n-2)$ be the fixed part of $\mathcal{H}_x^{q+1}(\Omega_x^p), \mathcal{H}_y^{q+1}(\Omega_Y^p)$ with respect to the action T; further define $H_X^{p,n-1}, H_Y^{p,n-2}$ as the dual spaces of $H_X^{n-p,0}, H_Y^{n-p-1,0}$ respectively. Then, as an application of Theorem 2, we have

Corollary 1. *There are canonical direct sum decompositions*

$$H^r(X \backslash x, C) = \sum_{p+q=r} H_X^{p,q}$$

$$H^r(Y \backslash y, C) = \sum_{p+q=r} H_Y^{p,q}$$

and natural exact sequences

$$0 \longrightarrow H^{n-1}(X\backslash x, C) \longrightarrow H^{n-1}(X \backslash Y, C) \longrightarrow H^{n-2}(Y \backslash y, C) \longrightarrow 0$$

$$0 \longrightarrow H^n(X\backslash x, C) \longrightarrow H^n(X \backslash Y, C) \longrightarrow H^{n-1}(Y \backslash y, C) \longrightarrow 0.$$

Remark. These direct sum decompositions also arise from the mixed Hodge structures of $X \backslash x$, $Y \backslash y$ in the sense of [1]. Thus, in this case, the characteristic function explains some aspect of the mixed

Hodge structure. Note also that this corollary shows the degeneracy of the spectral sequence $E_2^{p,q}(Y,y)=H^p(R^q\iota_*\iota^*\Omega_Y^\cdot)(\iota: Y\backslash y\hookrightarrow Y)$, and of $E_2^{p,q}(X,x)$ defined similarly.

Now we shall apply Theorem 1 and Theorem 2 to the study of algebraic manifolds. Let V be an n-dimensional submanifold in $P_{n+r}(\boldsymbol{C})$ and E the line bundle over V induced by the hyperplane sections. Since E^{-1} is negative, we can consider the quotient space $C(V)=E^{-1}/V$ by shrinking the zero section V into a point p, and we thus have isolated singularity $(C(V),p)$. Now we assume that V is the intersection of r non-singular hypersurfaces of $P_{n+r}(\boldsymbol{C})$ which are situated in a general position. Then $C(V)$ is a complete intersection, so it satisfies condition (L). (See [3].) Note that, on $C(V)$, there is the natural $\boldsymbol{C}^*$-action induced by the multiplication of $\boldsymbol{C}$ in the line bundle E^{-1}. We consider the functions $\chi^q_{C(V)}(t)$ $(0\leqq q\leqq n+2)$, $\chi_{C(V)}(s,t)$ to be defined with respect to this action. Now let $h^{p,q}(E^k)$ be the dimension of $H^q(V,\Omega^p(E^k))$ and let the polynomials $R^i(z_1,z_2,\cdots,z_i)$ $i=1,2,\cdots$ be defined inductively by $R^1(z)=(z-1)^{n+r+1}$, $R^{i+1}(z_1,z_2,\cdots,z_{i+1})=(z_1R^i(z_2,z_3,\cdots,z_{i+1})-z_2R^i(z_1,z_3,\cdots,z_{i+1}))/(z_2-z_1)$. Then we have

Corollary 2. *The assumptions being as above,* $H^q(V,\Omega^p(E^k))=0$ *if* $p+q\neq n$, $0<q<n$, $k\neq 0$; *further, if* $a_1,\cdots,a_r$ *are the degrees of the hypersurfaces defining* V, *then the following congruence holds*

$$\begin{aligned}\chi_{C(V)}(s,t)&\equiv\chi^0_{C(V)}(t)+s\chi^1_{C(V)}(t)+\sum_{p=1}^{n-1}s^{n-p+1}\Big(-\delta_{p,n-p}+\sum_{k\in\boldsymbol{Z}}h^{p,n-p}(E^k)t^k\Big)\\&\equiv\frac{t^{a_1}}{t^{a_1}-s}R^r\Big(\frac{t^{a_1}-1}{t-1},\cdots,\frac{t^{a_r}-1}{t-1}\Big)\\&\quad+s\sum_{j=1}^{r-1}\frac{\prod_{i=1}^{j}(1-t^{a_i})}{\prod_{i=1}^{j+1}(t^{a_i}-s)}R^{r-j}\Big(\frac{t^{a_{j+1}}-1}{t-1},\cdots,\frac{t^{a_r}-1}{t-1}\Big)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\bmod s^{n+1}\end{aligned}$$

where the last term should be regarded as a power series in s whose coefficients are rational functions in t. Moreover $\chi^1_{C(V)}(t)-\sum_{k<n}h^{n,0}(E^k)t^k$ *is a polynomial divisible by* t^n.

This corollary, combined with Theorems 22.1.1–22.1.2 of Hirzebruch [2], determines all of the dimensions of $H^q(V,\Omega^p(E^k))$.

The details will appear elsewhere.

References

[1] Deligne, P.: Theorie de Hodge. II. Publ. Math. I. H. E. S. No. 40, pp. 5–57 (1971).

[2] Hirzebruch, F.: Topological Methods in Algebraic Geometry. Springer Verlag (1966).

[3] Naruki, I.: A note on isolated singularity. I. Proc. Japan Acad., **51**, 317–319 (1975).