118. *Roots of Operators*

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We can completely determine nth roots of operators on Banach spaces when their spectra suit our convenience. One of the cases is given in Theorem 1. It is extremely connected with results by E. Hille (Theorems 1, 2 and 3 in [2]), those by J. G. Stampfli (Lemma 1 and Theorem 1 in [4]) and those by M. R. Embry (Thorems 3 and 4 in [1]). An application is an observation of the structure of periodic bounded automorphisms of Banach algebras. It is sammalized in Theorem 2.

1. Throughout this paper, we mean by an operator a bounded linear operator; n denotes a positive integer and $Sp(S)$ the spectrum of an operator S.

Theorem 1. *Suppose that S is an operator on a Banach space and that there exists on the plane a curve C leading from the origin 0 to the point at infinity such that $Sp(S)\cap C=\emptyset$. Then $\{z^{1/n} : 0\neq z\in C\}\cup\{0\}$, a union of n curves with an only common point 0, divides the plane n sectorial domains $D_0, \cdots, D_{n-2}$ and D_{n-1}, and it follows that*

(a) *for each D_k, there corresponds a unique nth root R_k of S such that $Sp(R_k)\subset D_k$; it necessarily is in the norm-closed algebra of operators generated by S and the identity operator I; and*

(b) *if T is an nth root of S, then it is of the form*

$$T=\sum_{k=0}^{n-1} R_k E_k,$$

where $E_0, \cdots, E_{n-2}$ and E_{n-1} are mutually orthogonal idempotent operators with $\sum_{k=0}^{n-1} E_k=I$, each of which commutes with S and hence with every R_k.

A part of the following proof is devoted to give the form of nth roots of I. It is not new; in fact, known by Stampfli in [3], but the way employed here is alternative and somewhat elementary.

Proof. Denote by f a branch of nth root function on the plane slit along the curve C, valued in D_0; and Γ a rectifiable Jordan contour having no common points with C, oriented in a counterclockwise direction. Define operators $R_0, \cdots, R_{n-2}$ and R_{n-1} by

$$R_0=f(S)=-\frac{1}{2\pi i}\int_\Gamma f(z)(S-zI)^{-1}dz,$$

and

$$R_k=\zeta_k R_0, \qquad k=1, \cdots, n-1,$$

where $\zeta_0=1$, $\zeta_1, \cdots, \zeta_{n-2}$ and ζ_{n-1} are the nth roots of unity. Then each R_k is an nth root of S with $Sp(R_k)\subset D_k$, putting $D_k=\zeta_k D_0$. By Mergelyan's theorem [3] on uniform polynomial approximations, f is a uniform limit of a sequence $\{p_j\}$ of polynomials on a compact set in which $Sp(S)$ and Γ are lying. So, $R_0=f(S)=\lim_{j\to\infty} p_j(S)$ is, and hence every R_k is, in the norm-closed algebra of operators generated by S and I.

Since T commutes with S, T commutes with R_0. Therefore, putting $T_0=R_0^{-1}T$, we have $T_0^n=I$ and hence $Sp(T_0)\subset\{\zeta_0, \cdots, \zeta_{n-1}\}$. Let γ be any circle with center ζ_k and radius sufficiently small, oriented in a counterclockwise direction. Then

$$E_k=-\frac{1}{2\pi i}\int_\gamma (T_0-zI)^{-1}dz$$

is an idempotent operator which commutes with S. Moreover, E_k's are mutually orthogonal and their sum is equal to I. Since for any $z\neq 0$,

$$\prod_{l=0}^{n-1}(T_0-z\zeta_l I)=z^n\prod_{l=0}^{n-1}\left(\frac{T_0}{z}-\zeta_l I\right)=z^n\left(\left(\frac{T_0}{z}\right)^n-I\right)=(1-z^n)I,$$

we have for any z distinct from every ζ_l,

$$(z-\zeta_k)(T_0-zI)^{-1}=-\frac{\prod_{l\neq 0}(T_0-z\zeta_l I)}{\prod_{l\neq k}(z-\zeta_l)}.$$

Thus we can find a $K>0$ such that for any z near to, but distinct from ζ_k, $\|(z-\zeta_k)(T_0-zI)^{-1}\|\leqq K$. Therefore, by the formula

$$(T_0-\zeta_k I)_{E_k}=-\frac{1}{2\pi i}\int_\gamma (z-\zeta_k)(T_0-zI)^{-1}dz,$$

we have $\|(T_0-\zeta_k I)E_k\|\leqq Kr$ with r the radius of the circle γ. Since the right side can be made arbitrarily small, we have $(T_0-\zeta_k)E_k=0$; and hence

$$T=R_0\sum_{k=0}^{n-1}T_0E_k=\sum_{k=0}^{n-1}\zeta_k R_0E_k=\sum_{k=0}^{n-1}R_kE_k.$$

Suppose next that R is an nth root of S with $Sp(R)\subset D_k$. Then, R must be of the form $R=\sum_{l=0}^{n-1}R_lE_l$ and hence, it must be in the norm-closed algebra $\mathfrak{A}$ of operators generated by $R_0, E_0, \cdots, E_{n-2}$ and E_{n-1}, which is necessarily abelian. Given any non-zero multiplicative linear functional ϕ on $\mathfrak{A}$, $\phi(E_l)$'s are 0 with only one exception $\phi(E_m)$ which must be 1. Thus, $D_m\ni\phi(R_m)=\sum_{l=0}^{n-1}\phi(R_l)\phi(E_l)=\phi(R)\in D_k$. It follows that $m=k$; and $\phi(E_l)=0$, whenever $l\neq k$. Therefore, E_l is in the radical of $\mathfrak{A}$ and hence it must be 0 whenever $l\neq k$. Consequently, we know that $E_k=I$ and that $R=R_k$. Now the proof is completed.

2. Two remarks to Theorem 1 are given here.

In order that every E_k is an image of S by an analytic function applicable to S, it is sufficient that for T the following condition employed by Hille in [2] and Embry in [1] is satisfied:

$$Sp(T)\cap\zeta_k Sp(T)=\emptyset,\qquad k=1,\cdots,n-1.$$

In fact, a function f_k taking 1 on a compact set containing $\Delta_k=\{z\in Sp(S): z^n\in D_k\cap Sp(T)\}$, and taking 0 on a compact set containing $Sp(S)\backslash\Delta_k$, but being disjoint with the former, is applicable to S and $f_k(S)$ is nothing but E_k. This observation leads us to the main theorem in [1], which says that under this condition every operator which commutes with S commutes with T.

In order that every E_k is, and hence T is, in the norm-closed algebra of operators generated by S and I, it is sufficient that for T the above-mentioned condition and the following one are satisfied:

"$Sp(T)$ has no holes (i.e., no bounded complementary connected components) which have common points with $\zeta_k Sp(T)$ for some $k=1,\cdots,n-1$."

In fact, by Mergelyan's theorem [3], f_k defined above is approximated uniformly by a sequence of polynomials; therefore, E_k is approximated by polynomials of which variable is S.

3. A consequence of Theorem 1 is the spectral decomposition for periodic bounded automorphisms of Banach algebras. It is stated as follows:

Theorem 2. *Let A be a Banach algebra and ρ an operator on A. Then, ρ is an automorphism of period n if and only if there exist linearly independent subspaces $A_{\zeta_0},\cdots,A_{\zeta_{n-2}}$ and $A_{\zeta_{n-1}}$ of A indexed by the nth roots of unity, such that A is spanned by them and for them the following is satisfied:*

$$A_{\zeta_k}A_{\zeta_l}\subset A_{\zeta_k\zeta_l}\text{ and }\rho(x)=\zeta_k x\qquad\text{for } x\in A_{\zeta_k};\ k,l=0,\cdots,n-1.$$

In the case, each A_{ζ_k} is a closed subspace of A.

Let in particular A be an involutive Banach algebra. Then, ρ is a $*$-automorphism of period n if and only if in addition the following is satisfied:

$$(A_{\zeta_k})^*\subset A_{\bar{\zeta}_k},\qquad k=0,\cdots,n-1.$$

Proof. By Theorem 1, we know that ρ has the form $\rho=\sum_{k=0}^{n-1}\zeta_k\varepsilon_k$, where ε_k's are mutually orthogonal idempotent operators on A such that their sum is equal to the identity operator on A. Denote by A_{ζ_k} the range of ε_k. Then A_{ζ_k}'s are linearly independent and span A; and moreover, we can see that $A_{\zeta_k}=\{x\in A:\rho(x)=\zeta_k x\}$, $k=0,\cdots,n-1$. This is enough to show the necessities. The sufficiencies are easily shown and the proof is completed.

References

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