107. *Limit Theorems for Poisson Branching Processes*

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1. The process treated here is a model of the population growth in a biological system in which each object gives births at various times of its life length and new born objects behave as their parents independently of others. The process is specified by two nonnegative continuous functions on $[0, \infty)$ $\lambda(x)$, $\mu(x)$ and a probability generating function $h(s)=\sum_{n=1}^{\infty} h_n s^n$, $\sum_{n=1}^{\infty} h_n=1$, $h_n \geqq 0$ $(n=1,2,\cdots)$; *a living object of age x gives births to j objects before it reaches age $x+dx$ without dying itself with a probability $h_j\lambda(x)dx$ and dies before age $x+dx$ with a probability $\mu(x)dx$* where these probabilities are independent of each other and of past history. This process appeared in [2] as a special case of general age dependent branching processes and was called a *Poisson branching process.* In this paper limit theorems will be given for probability generating functions of the population size at time t of Poisson branching processes. Limit theorems of such type are studied by Ryan [5] for subcritical general age dependent branching processes. His results contain a part of ours as a special case. The forms and proofs of theorems given here are simpler than Ryan's and almost parallel with ones of age dependent branching processes given in [1].

2. Let $Z(t)$ be the population size at time t of a Poisson branching process specified by $\lambda(x)$, $\mu(x)$ and $h(s)$ as in the first section and let $F(s,t)$ be its generating function; $F(s,t)=E[s^{Z(t)}]$, $0\leqq s\leqq 1$. We always assume that the process starts with a single object of age 0. Let L be the time when the initial object dies and $G(t)$ be the distribution function of L; $G(t)=\int_0^t \mu(u)\exp\left(-\int_0^u \mu(r)dr\right)du$. By conditioning on L we get

$$F(s,t)=s(1-G(t))E\left[\exp\left\{\int_0^t \log F(s,t-u)dN(u)\right\}\middle| L>t\right] + \int_0^t E\left[\exp\left\{\int_0^u \log F(s,t-v)dN(v)\right\}\middle| L=u\right]dG(u),$$

in which we denote by $N(t)$ the number of direct children of the initial particle that have been ever born untill time t. Then we have

$$F(s,t)=s(1-G(t))\exp\left\{\int_0^t (h(F(s,t-u))-1)\lambda(u)du\right\} + \int_0^t \exp\left\{\int_0^u (h(F(s,t-v))-1)\lambda(v)dv\right\}dG(u). \tag{1}$$

When $m=h'(1-)<\infty$, $F(s, t)$ is continuous in $t\in[0,\infty)$ for each $s\in[0,1]$ and is the unique solution of (1) with $0\leqq F\leqq 1$ (see [3] for the proof).

From now on we assume $m=h'(1-)<\infty$.

Put $g(s)=\int_0^\infty \exp\left(\int_0^t \lambda(u)du(h(s)-1)\right)dG(t)$ and let q be the smallest root in $[0,1]$ of $g(s)=s$. Then q is the extinction probability for $Z(t)$; $q=\lim_{t\to\infty} F(0,t)$ ([3]), and $q=1$ is equivalent to $g'(1)<1$ and $g(1)=1$. We call our process *subcritical* if $g'(1)<1$ and $g(1)=1$, *critical* if $g'(1)=g(1)=1$ and *supercritical* if $g'(1)<1$ or $g(1)<1$.

Let $M(t)$ denote $E[Z(t)]=F'(1,t)$. From (1) it follows that

$$M(t)=m\int_0^t M(t-u)\lambda(u)(1-G(u))du+(1-G(t)).$$

We can see that $M(t)$ is bounded on each finite interval and the standard renewal theorem deduces ([5], [3]) the following.

Lemma 1. *Suppose there exists* α *such that* $m\int_0^\infty \exp(-\alpha t)(1-G(t))\lambda(t)dt=1$ *and* $\int_0^\infty \exp(-\alpha t)(1-G(t))dt<\infty$. *Then* $M(t)\sim a\cdot\exp(\alpha t)$ *as* $t\to\infty$ *where* $a=\int_0^\infty \exp(-\alpha t)(1-G(t))dt\left(m\int_0^\infty \exp(-\alpha t)(1-G(t))\lambda(t)t dt\right)^{-1}$.

3. In this section we study the asymptotic behavior of $F(s,t)$ as $t\to\infty$ for subcritical processes under the condition of Lemma 1. Let α be a number in Lemma 1. Note that α is necessarily negative when $g(1)=1$ and $g'(1)<1$.

Lemma 2. *If* $g'(1)<1$, $g(1)=1$ *and the condition of Lemma* 1 *is satisfied, then* $\sup_{t>0,1\geqq s\geqq 0}\exp(-\alpha t)(1-F(s,t))\equiv K<\infty$.

Before going into the proof of the lemma we state the main theorem.

Theorem 1. *If* $g'(1)<1$, $g(1)=1$, *the condition of Lemma* 1 *is satisfied and*

$$\int_0^\infty te^{-\alpha t}\lambda(t)(1-G(t))dt<\infty, \tag{2}$$

then $\liminf_{t\to\infty}(1-F(s,t))\exp(-\alpha t)>0$ *iff* $\sum h_j(j\log j)<\infty$ *and* $E[X\log X]<\infty$, *where* $X\equiv\int_0^L \exp(-\alpha t)\lambda(t)dt$. *In this case* $\lim_{t\to\infty}(1-F(s,t))\exp(-\alpha t)\equiv Q(s)$ *exists and defines a positive analytic function of* $s\in[0,1)$ *with* $Q'(1-)<\infty$.

Remark 1. We can see that $\sum h_j(j\log j)<\infty$ and $E[X\log X]<\infty$ iff $E[Y\log Y]<\infty$ where $Y\equiv\int_0^L \exp(-\alpha t)dN(t)$. Therefore, by the inequality

$$E[Y \log Y] \geqq E[E[Y \mid N(L)] \log E[Y \mid N(L)]]$$
$$\geqq \sum_{k=1}^{\infty} p_k k \log k \int_0^{\infty} \exp(-\alpha t) dL_k(t)$$

where $p_k = P[N(L)=k]$ and $L_k(t) = k^{-1} E[N(t) \mid N(L)=k]$, the sufficiency part of the theorem is reduced to the result of [5].

The next theorem is an immediate corollary of Theorem 1.

Theorem 2. *Suppose that conditions of Theorem 1 is satisfied and that $E[X \log X]$ and $\sum h_j(j \log j)$ are both finite where X is defined in Theorem 1. Then $\lim_{t\to\infty} P[Z(t)=k \mid Z(t)>0] = b_k$ exist and $\{b_k\}_{k=1}^{\infty}$ is a probability distribution with mean $\sum kb_k = Q'(1-)(Q(0))^{-1}$.*

Now we prove Lemma 2. After simple calculations (1) is rewritten as follows;

$$(3) \quad \begin{aligned} 1-F(s,t) &= \xi^1(t) - \xi^2(t) - \xi^3(t) - \xi^4(t) \\ &\quad + m\int_0^t (1-F(s,t-u))\lambda(u)(1-G(u))du \end{aligned}$$

where

$$\xi^1(t) = (1-G(t)) \exp\left\{-\int_0^t (1-h(F(s,\,t-u)))\lambda(u)du\right\},$$
$$\xi^2(t) = (1-G(t)) A\left(\int_0^t (1-h(F(s,t-u)))\lambda(u)du\right),$$
$$\xi^3(t) = \int_0^t A\left(\int_0^u (1-h(F(s,t-v)))\lambda(v)dv\right) dG(u),$$
$$\xi^4(t) = \int_0^t \{m(1-F(s,t-u)) - (1-h(F(s,t-u)))\}\lambda(u)(1-G(u))du,$$

with $A(x) = x - 1 + \exp(-x)$. Let us write $\xi^1_\alpha(t) = \exp(-\alpha t)\xi^1(t)$ etc. Put $H(t) = 1-F(s,t)$ and $R_\alpha(t) = \int_0^t m \exp(-\alpha u)\lambda(u)(1-G(u))du$. Then $H_\alpha(t) \leqq \exp(-\alpha t)(1-G(t)) + \int_0^t H_\alpha(t-u)dR_\alpha(u)$ since ξ^2, ξ^3 and ξ^4 are all non-negative. Lemma 2 now follows from the Renewal theorem ([4]).

For the proof of Theorem 1 we need the following lemmas.

Lemma 3. *$\xi^2(t)$, $\xi^3(t)$ and $\xi^4(t)$ are all Riemann integrable.*

Proof. Taking Laplace transforms, it follows from (3) that

$$\hat{H}_\alpha(x)(1-\hat{R}_\alpha(x)) = \hat{\xi}^1_\alpha(x) - \hat{\xi}^2_\alpha(x) - \hat{\xi}^3_\alpha(x) - \hat{\xi}^4_\alpha(x)$$

where we set $\hat{\xi}^1_\alpha(x) = \int_0^\infty \exp(-xt)\xi^1_\alpha(t)dt$ etc. Since $\hat{\xi}^1_\alpha(0+) < \infty$, by compairing the signs of terms in both sides, we see that $\hat{\xi}^i_\alpha(0+) < \infty$, $i=2,3,4$.

The next lemma furnishes a key for the proof of Theorem 1.

Lemma 4. *Let Y be a non-negative random variable with $E[Y] = 1$. Then for any $\delta > 0$ $\int_0^\delta E[A(uY)]u^{-2}du < \infty$ iff $E[Y \log Y] < \infty$.*

Corollary. *Let $f(s) = \sum_{i=0}^{\infty} q_i s^i$ be a probability generating function with $c = f'(1) < \infty$. Then for $0<\delta<1$, $\int_0^\delta [c - u^{-1}(1-f(1-u))]u^{-1}du$*

$<\infty$ *iff* $\sum_{i=0}^{\infty} q_i(i \log i)<\infty$.

We omit the proof of these results (see [1] for a proof).

Proof of Theorem 1. Let $\liminf (1-F(0,t)) \exp(-\alpha t)>0$. We first note that there exists a positive constant C such that for all $t\geqq 0$

$$(1-F(0,t))>Ce^{\alpha t} \quad \text{and} \quad 1-h(F(0,t))>Ce^{\alpha t}.$$

By Lemma 3, using the inequality $A(x)>x-1$, $x>0$,

$$\infty>\int_0^\infty \xi_\alpha^2(t)dt$$
$$>C\int_0^\infty (1-G(t))dt\int_0^t e^{-\alpha v}\lambda(v)dv-\int_0^\infty e^{-\alpha t}(1-G(t))dt,$$

and then the hypothesis of the theorem implies

$$\int_0^\infty (1-G(t))dt\int_0^t e^{-\alpha v}\lambda(v)dv<\infty. \tag{4}$$

From (2) and (4) it follows

$$\int_0^\infty dt\int_t^\infty dG(u)\int_0^u e^{-\alpha v}\lambda(v)dv<\infty. \tag{5}$$

We see similarly that

$$\infty>\int_0^\infty \xi_\alpha^3(t)dt>\int_0^\infty e^{-\alpha t}dt\int_0^t A\left(C\int_0^u e^{\alpha(t-v)}\lambda(v)dv\right)dG(u).$$

Since $A(x)<x$, this inequality, combined with (5), leads to

$$\infty>\int_0^\infty e^{-\alpha t}dt\int_0^\infty A\left(C\int_0^u e^{\alpha(t-v)}\lambda(v)dv\right)dG(u)$$
$$=(-\alpha)^{-1}C\int_0^C E[A(uX)]u^{-2}du.$$

Consequently, by Lemma 4, we obtain $E[X\log X]<\infty$ since $E[X]=1$.

We deduce in a similar way, using Corollary instead of Lemma 4 and using integrability of ξ_α^4, that $\sum h_j(j\log j)<\infty$.

To prove the converse part, we assume that $E[X\log X]<\infty$ and $\sum h_j(j\log j)<\infty$. Since $E[X\log X]<\infty$ implies (4), we see that $\xi_\alpha^2(t)$ is directly Riemann integrable. ξ_α^1, ξ_α^3 and ξ_α^4 are also directly Riemann integrable: for example

$$\sup_{n\leqq t<n+1}\xi_\alpha^3(t)<e^{-\alpha}\inf_{n-1<u<n}\xi_\alpha^3(u)+Km\int_{n-1}^{n+1}dG(u)\int_0^u e^{-\alpha v}\lambda(v)dv$$

and the sum of the right hand sides over n converges. The renewal theorem therefore can be applied and then

$$\lim_{t\to\infty} e^{-\alpha t}(1-F(s,t))=\frac{\int_0^\infty(\xi_\alpha^1(t)-\xi_\alpha^2(t)-\xi_\alpha^3(t)-\xi_\alpha^4(t))dt}{\int_0^\infty tdR_\alpha(t)}$$

exists. We denote this limit by $Q(s)$. Now we claim that

$$\lim_{s\uparrow 1}\frac{Q(s)}{1-s}=\frac{\int_0^\infty(1-G(t))e^{-\alpha t}dt}{\int_0^\infty tdR_\alpha(t)}. \tag{6}$$

If we prove this equation, since $Q(s)$ is non-increasing, we obtain that

$Q(s)>0$, $0\leqq s<1$ and the proof is completed.

For the proof of (6) it suffices to show that

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty (\xi_\alpha^2(t)+\xi_\alpha^3(t)+\xi_\alpha^4(t))dt=0.$$

If we take a convention such that $F(s,t)=1$ for $t<0$, then $(1-s)^{-1}(\xi_\alpha^2(t)+\xi_\alpha^3(t))$ is written as

$$\int_0^\infty\int_0^u \frac{1-h(F(s,t-v))}{1-s}\lambda(v)dv \times\left[1-\frac{1-\exp\left(\int_0^u (1-h(F(s,t-v)))\lambda(v)dv\right)}{\int_0^u (1-h(F(s,t-v)))\lambda(v)dv}\right]dG(u).$$

From Lemma 1 and Lemma 2 it follows that

$$\frac{(\xi_\alpha^2(t)+\xi_\alpha^3(t))}{1-s}<\text{const.}e^{-\alpha t}E[A(mKe^{\alpha t}X)].$$

Since the right hand side of this inequality is integrable on $[0,\infty)$ by Lemma 4, the dominated convergence theorem is now applied to obtain

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty (\xi_\alpha^2(t))+\xi_\alpha^3(t))dt=0.$$

We can argue similarly to get that $\sum h_j(j\log j)<\infty$ implies

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty \xi_\alpha^4(t)dt=0.$$

Thus the theorem is proved.

Remark 2. Evaluation (4) is not implied by conditions of Theorem 1, i.e. there exists a triple $\lambda(t)$, $\mu(t)$ and $h(s)$ for which conditions of Theorem 1 are satisfied but (4) fails to hold.

4. For the supercritical processes we get only an unsatisfactory result.

Theorem 3. *Let* $g'(1-)>1$. *If there exists a number* β *such that* $1=\gamma\int_0^\infty \exp(-\beta t)\lambda(t)(q-J(t))dt$ *and* $\int_0^\infty \exp(-\beta t)(q-J(t))dt<\infty$, *where* $J(t)=\int_0^t \exp\left\{(h(q)-1)\int_0^u \lambda(v)dv\right\}dG(u)$ *and* $\gamma=h'(q)$, *then* $(q-F(0,t))\exp(-\beta t)$ *is bounded on* $[0,\infty)$.

For the proof the same method as in the proof of Lemma 2 is applied.

In order to demonstrate that a number β defined in the above will be proper one, we give a simple example. If we take $\lambda(t)\equiv\lambda$, $\mu(t)\equiv\mu$ where λ and μ are positive constants, then our process is a Markov branching process determined by the backward equation $\frac{\partial}{\partial t}F(s,t)=u(F(s,t))$, $u(s)=(\lambda+\mu)\left(\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}s\cdot h(s)-s\right)$ with $\beta=\gamma\lambda q-\mu q^{-1}$

which coincides with an usual parameter β_0 determined by the equation $1=u'(q)\int_0^\infty \exp(-(\lambda+\mu+\beta_0)t)dt$.

References

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