155. On the Fundamental Solution of a Degenerate Parabolic System

By Kenzo SHINKAI
University of Osaka Prefecture

(Comm. by Kôsaku YOSIDA, M. J. A., Oct. 13, 1975)

Introduction. In the recent paper [2], the author has shown that the method used in C. Tsutsumi [3] to construct the pseudo-differential symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator is applicable to some parabolic systems. The purpose of the present paper is to show that the above method is also applicable to a parabolic system which degenerates at $t=0$. As an application we construct in § 2 the pseudo-differential symbol of the fundamental solution of a degenerate parabolic operator of higher order which includes the operator treated by M. Miyake [1]. In the following the notation of [2] will be freely used.

1. The fundamental solution of a degenerate system. Let us consider the Cauchy problem for a system of pseudo-differential equations

$$\partial_t u(t,x)+p(t;X,D_x)u(t,x)=0, \tag{1}$$

$$\lim_{s\searrow 0} u(t,u)=u_0(x), \tag{2}$$

where $p(t;x,\xi)\in\mathcal{E}_t^0(S^m_{\rho,\delta})$, $0\le\delta<\rho\le 1$. We denote by $z(t,s;x,\xi)$ an $M\times M$ matrix of symbols which satisfies $\partial_t z(t,s;x,\xi)+p(t;x,\xi)z(t,s;x,\xi)=0$, $z(s,s;x,\xi)=I$, where I denotes the identity matrix. We denote by $|p|$ the norm of an $M\times M$ matrix p, i.e., $p=\sup\{|py|/|y|;\,0\ne y\in \boldsymbol{C}^M\}$.

Definition. We say that a system of pseudo-differential operators $\partial_t+p(t;X,D_x)$ satisfies the property (F), when there exists a non-negative continuous function $\lambda(t;x,\xi)$ and following two conditions are satisfied:

i) For any α,β there exists a constant $C_{\alpha,\beta}$ such that

$$\int_s^t |p^{(\alpha)}_{(\beta)}(\sigma;x,\xi)|\,d\sigma\le C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\left\{\int_s^t\lambda(\sigma;x,\xi)d\sigma+1\right\} \tag{3}$$

for $0\le s\le t\le T$.

ii) There exist constants $d>0$ and $C>0$ such that

$$|z(t,s;x,\xi)|\le C\exp\left[-d\int_s^t\lambda(\sigma;x,\xi)d\sigma\right]\qquad\text{for } 0\le s\le t\le T. \tag{4}$$

When a system $\partial_t+p(t;X,D_x)$ is parabolic in the sense of Petrowskii, it satisfies the property (F) with $\lambda(t;x,\xi)=\langle\xi\rangle^m$ in any finite layer $[0,T]\times R^{2n}_{x,\xi}$. But in the case of degenerate $p(t;x,\xi)$, we must choose a degenerate $\lambda(t;x,\xi)$. Here we give a class of systems for which the property (F) is easily verified.

Lemma 1. *Let $p(t; x, \xi)=f(t)\tilde{p}(t; x, \xi)$ where $f(t)$ is a non-negative continuous function and $\tilde{p}(t; x, \xi) \in \mathcal{E}_t^0(S_{\rho,\delta}^m)$ is a matrix of which the real part of every eigenvalue is not less than $d\langle\xi\rangle^m$. Then $\partial_t+p(t; X, D_x)$ fulfills the property (F) with $\lambda(t; x, \xi)=f(t)\langle\xi\rangle^m$.*

The property (F) is stable in the following sense:

Lemma 2. *Let $\partial_t+p(t; X, D_x)$ satisfy the property (F), and let $q(t; x, \xi) \in \mathcal{E}_t^0(S_{\rho,\delta}^m)$. If there exist constants C_1 and C_2 such that $0<C_1<d/C$ and*

$$\int_s^t |q(\sigma; x, \xi)| d\sigma \leq C_1 \int_s^t \lambda(\sigma; x, \xi) d\sigma + C_2, \tag{5}$$

then $\partial_t+p(t; X, D_x)+q(t; X, D_x)$ also satisfies the property (F).

Theorem 1. *Let $\partial_t+p(t; X, D_x)$ satisfy the property (F). Then there exists the symbol of fundamental matrix $e(t, s; x, \xi) \in w\text{-}\mathcal{E}_{t,s}^0(S_{\rho,\delta}^0)$ for the Cauchy problem (1), (2), i.e., $e(t, s; x, \xi)$ satisfies*

$$\partial_t e(t, s; x, \xi)+p(t)\circ e(t, s)(x, \xi)=0, \tag{6}$$

$$w\text{-}\lim_{t \downarrow s} e(t, s; x, \xi)=I. \tag{7}$$

The proof of the theorem is the same with that of Theorem 1 and Theorem 2 in [2].

2. The fundamental solution of a degenerate parabolic operator of higher order. Let

$$L=\partial_t^M+a_1(t; X, D_x)\partial_t^{M-1}+\cdots+a_M(t; X, D_x), \tag{8}$$

where

$$a_j(t; x, \xi)=\textstyle\sum_{k=0}^j a_{j,k}(t; x, \xi)t^{jl-k}, \qquad j=1, 2, \cdots, M. \tag{9}$$

Then we have the following

Theorem 2. *Let L satisfy the following conditions:*

i) $a_{j,k}(t; x, \xi) \in \mathcal{E}_t^0(S_{\rho,\delta}^{m(j,k)})$ *where*

$$m(j, k)=jm-km/(l+1). \tag{10}$$

ii) *For a positive constant d, every root $\tau_j(t; x, \xi)$ of the equation*

$$\tau^M+a_{1,0}(t; x, \xi)\tau^{M-1}+\cdots+a_{M,0}(t; x, \xi)=0$$

satisfies

$$\mathrm{Re}\,\tau_j(t; x, \xi) \leq -d\langle\xi\rangle^m, \qquad j=1, 2, \cdots, M. \tag{11}$$

Then there exist pseudo-differential symbols $g_j(t, s; x, \xi) \in w\text{-}\mathcal{E}_{t,s}^0(S_{\rho\delta}^{-jm/(l+1)})$, $j=0, 1, \cdots, M-1$, such that for $\psi_j(x) \in \mathcal{B}$, $j=0, 1, \cdots, M-1$, the function

$$v(t, x)=Os\text{-}[e^{-iy\xi}g_j(t, 0; x, \xi)\psi_j(x+y)] \tag{12}$$

is a solution of the following Cauchy problem

$$Lv=0, \tag{13}$$

$$\partial_t^j v(0, x)=\psi_j(x), \qquad j=0, 1, \cdots, M-1. \tag{14}$$

Proof. Set

$$h(t; \xi)=t^l\langle\xi\rangle^m+\langle\xi\rangle^{m/(l+1)}, \tag{15}$$

and let

(16) $$p(t;x,\xi)=t^l\langle\xi\rangle^m\begin{bmatrix}0 & ,-1, & & \\ & & ,0 & ,-1 \\ a_{M,0}\langle\xi\rangle^{-Mm}, & \cdots, & a_{2,0}\langle\xi\rangle^{-2m}, & a_{1,0}\langle\xi\rangle^{-m}\end{bmatrix},$$

(17) $$q(t;x,\xi)=\begin{bmatrix}0,-\langle\xi\rangle^{m/(l+1)}, & & \\ & & 0,-\langle\xi\rangle^{m/(l+1)} \\ q_M, q_{M-1} & ,\cdots, & q_2, q_1\end{bmatrix},$$

where $q_j(t;x,\xi)$, $j=1,2,\cdots,M$, are decided so that if

(18) $$\begin{cases}[\partial_t+p(t;X,D_x)+q(t;X,D_x)]u(t,x)=0,\\ u_1(t,x)=h(t;D_x)^{M-1}v(t,x),\end{cases}$$

then v satisfies $Lv=0$. Then we can prove that $\partial_t+p(t;X,D_x)+q(t;X,D_x)$ satisfies the property (F), and we have the fundamental matrix $e(t,s;x,\xi)$ of the operator $\partial_t+p(t;X,D_x)+q(t;X,D_x)$. Let $b_{j,k}(t;\xi)$ be the symbols which are decided by

(19) $$\begin{cases}b_{1,1}(t;\xi)=h(t;\xi)^{M-1},\\ b_{j,k}(t;\xi)=h(t;\xi)^{-1}\{b_{j-1,k-1}(t;\xi)+\partial_t b_{j-1,k}(t;\xi)\},\\ \qquad\qquad j=2,\cdots,M;\ k=1,\cdots M,\\ b_{j,k}(t;\xi)=0 \qquad \text{when } j<k \text{ or } k=0.\end{cases}$$

Then $g_0, g_1, \cdots, g_{M-1}$ are given by the equations:

(20) $$g_j(t,s;x,\xi)=\sum_{k=1}^M h(t)^{1-M}\circ e_{1,k}(t,s)\circ b_{k,j+1}(s)(x,\xi).$$

References

[1] Miyake, M.: Hypoelliptic degenerate evolution equations of the second order (to appear).

[2] Shinkai, K.: On symbols of fundamental solutions of parabolic systems. Proc. Japan Acad., **50**, 337–341 (1974).

[3] Tsutsumi, C.: The fundamental solution for a degenerate parabolic pseudo-differential operator. Proc. Japan Acad., **50**, 11–15 (1974).