145. *Eisenstein Integrals and Singular Cauchy Problems*

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1. The classical Euler-Poisson-Darboux (EPD) equations of Weinstein (see e.g. [15]), and various formulas arising in their solution, are known to possess group theoretic content, and various other analogous classes of singular Cauchy problems also have been studied from this point of view (cf. [4]–[6], [11]). We will discuss here some aspects of the general situation in the context of harmonic analysis on symmetric spaces (cf. [7]–[10], [12]–[14] for notation). Thus let G be a real connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup so that $V=G/K$ is a symmetric space of noncompact type. Let $g^{\sim}=k^{\sim}+p$ be a Cartan decomposition, $a\subset p$ a maximal abelian subspace, and we will suppose that $\dim a=\text{rank } V=1$. Let $G=KAN$ denote the related Iwasawa decomposition with components $g=k(g)\exp H(g)n(g)$ and write g_{λ} for the standard root subspaces in $g^{\sim}$ (here we have positive roots α and possibly 2α). Set $\rho=(1/2)\sum m_{\lambda}\lambda$ for $\lambda>0$ where $m_{\lambda}=\dim g_{\lambda}$ and pick an element $H_0\in a$ with $\alpha(H_0)=1$ while setting $a_t=\exp tH_0$; for $\mu\in \boldsymbol{R}\approx a^*$ we put $\mu(tH_0)=\mu t$ and then $\rho=1/2m_{\alpha}+m_{2\alpha}$. We identify $(0,\infty)$ with a Weyl chamber $a_+\subset a$. Let M (resp. M') be the centralizer (resp. normalizer) of $A=\exp a$ in K so that the Weyl group (of order $w=2$) is $W=M'/M$ and the boundary of V is $B=K/M$.

Given now $v=gK\in V$ and $b=kM\in B$ one writes $A(v,b)=-H(g^{-1}k)$ and the Fourier transform of $f\in L^2(V)$ is defined by

$$\tilde{f}(\mu,b)=\int_V f(v)e^{(i\mu+\rho)A(v,b)}dv \tag{1.1}$$

for $\mu\in a^*$ and $b\in B$. The inversion formula is

$$f(v)=\frac{1}{w}\int_{a^*\times B}\tilde{f}(\mu,b)e^{(-i\mu+\rho)A(v,b)}|c(\mu)|^{-2}d\mu db \tag{1.2}$$

where $c(\mu)$ is the standard Harish-Chandra function (and $w=2$). Now $a^*/W\approx a^*_+$ and one can write

$$L^2(V)=\int_{a^*/W}\mathcal{H}_{\mu}|c(\mu)|^{-2}d\mu \tag{1.3}$$

$$\mathcal{H}_{\mu}=\left\{\phi_{\mu}(v)=\int_B e^{(-i\mu+\rho)A(v,b)}\varphi(b)db\right\} \tag{1.4}$$

for $\varphi\in L^2(B)$. The quasiregular representation of G on $L^2(V)$, defined by $L(g)f(v)=f(g^{-1}v)$, decomposes in the form $L=\int_{a^*/W} L_\mu |c(\mu)|^{-2}d\mu$ where L_μ acts in $\mathcal{H}_\mu$ by the same rule as L with L_μ irreducible and unitary.

We recall here also the definition of the mean value of a function f over the orbit of $g\pi(h)=gu$ under the isotropy subgroup $I_v=gKg^{-1}$ at $v=\pi(g)$ ($\pi: G\to G/K$ is the canonical map). Thus, noting that $M^h f \equiv M^u f$, one can write

$$(1.5)\qquad (M^h f)(v)=\int_K f(gk\pi(h))dk=F(u,v)$$

and the so called Darboux equation is $D_u F=D_v F=(M^h(Df))(v)$ where $D\in D(G/K)$. The zonal spherical functions on G are defined by

$$(1.6)\qquad \tilde{\varphi}_\mu(g)=\int_K e^{(i\mu-\rho)H(gk)}dk$$

for $\mu\in a^*$ and one can evidently write $\tilde{\varphi}_\mu(g)=\varphi_\mu(gK)$ where it is known that $\tilde{\varphi}_{-\mu}(g^{-1})=\tilde{\varphi}_\mu(g)$. It is easy to show that the Fourier transform of $M^h\equiv M^u\in\mathcal{E}'(V)$ is $\mathcal{F}M^h=\tilde{\varphi}_\mu(h)$. We mention also that there are natural polar coordinates in a dense submanifold of V arising from the decomposition $G=K\bar{A}_+K$, $A_+=\exp a_+$, provided by the diffeomorphism $(kM,a)\to kaK: B\times A_+\to V$. Thus the polar coordinates of $\pi(g)=\pi(k_1 a k_2)\in V$ are (k_1M,a). Further if $h=\tilde{k}a\hat{k}$ with $a\in A_+$ then $(M^h f)(v)\equiv(M^u f)(v)=(M^a f)(v)$.

2. The objects of interest in a generalized *EPD* theory are the radial components of a basis for the $\mathcal{H}_\mu$ spaces of (1.4), multiplied by a suitable weight function. Now $D(G/K)$ is generated by a single Laplacian Δ and we look at the radial component Δ_R of Δ, passing this from the coordinate t in $a_t\in A$ to $\pi(A)$ in an obvious manner, and setting $M_t=M_{a_t}$ with $\mathcal{F}M_t=\tilde{\varphi}_\mu(a_t)$, one has an eigenvalue equation (cf. [14])

$$(2.1)\qquad \begin{aligned}&[D_t^2+(m_\alpha+m_{2\alpha})\coth tD_t+m_{2\alpha}\,\mathrm{th}\,tD_t]\tilde{\varphi}_\mu\\ &\quad+[\mu^2+((1/2)m_\alpha+m_{2\alpha})^2]\tilde{\varphi}_\mu=0\end{aligned}$$

where $D_t=d/dt$ and th$=$tanh. The solution of (2.1), "nice" at $t=0$, is

$$(2.2)\qquad \hat{R}^0(t,\mu)=\tilde{\varphi}_\mu(\exp tH_0)=F(\delta,\beta,\gamma-\mathrm{sh}^2t)$$

where $\delta=(1/4)(m_\alpha+2m_{2\alpha}+2i\mu)$, $\beta=(1/4)(m_\alpha+2m_{2\alpha}-2i\mu)$, and $\gamma=(1/2)(m_\alpha+m_{2\alpha}+1)$. The idea now is to embed $\hat{R}^0(t,\mu)$ in a "canonical" sequence of "resolvants" $\hat{R}^m(t,\mu)$ (m could be a multi-index) such that the resolvant initial conditions $\hat{R}^m(0,\mu)=1$ and $\hat{R}^m_t(0,\mu)=0$ are satisfied while the associated singular differential equations for the $\hat{R}^m$ are "split" by certain recursion relations as indicated below.

First we recall that a basis for $L^2(B)$ can be taken in the form of functions $kM\to>\langle w_i^\tau,\pi_\tau(k)w_1^\tau\rangle_\tau$, $1\le i\le d(\tau)$, where $\tau=(\pi_\tau,V_\tau)$ (with $\dim V_\tau=d(\tau)$) runs over the set T of inequivalent irreducible unitary representations of K such that $\dim V_\tau^M=1$ ($V_\tau^M\subset V_\tau$ is the set of elements fixed by M). Here one knows $\dim V_\tau^M=1$ or 0 and w_1^τ is a basis vector for V_τ^M with $\{w_i^\tau\}$, $1\le i\le d(\tau)$, and orthonormal basis for V_τ under a scalar product $\langle\ ,\ \rangle_\tau$. These representations can be parameterized as follows (see [9] for references). If $m_{2\alpha}=0$, $T\sim\{(p,q)\}$ $q=0$; if $m_{2\alpha}=1$, $T\sim\{(p,q)\}$ with $p\in Z_+$ and with $(p,q)\in Z_+\times Z$ where $p\pm q\in 2Z_+$; and if $m_{2\alpha}=3$ or 7, $T\sim\{(p,q)\}$ with $(p,q)\in Z_+\times Z_+$ where $p\pm q\in 2Z_+$. The proof of the following theorem results from [9].

Theorem 1. *The radial components of basis vectors in $\mathcal{H}_\mu$ can be expressed through Eisenstein integrals in the form*

$$\begin{aligned}\psi_{-u,\tau}(a_tK)&=\int_K e^{(i\mu-\rho)H(a_t^{-1}k)}\langle w_1^\tau,\pi_\tau(k)w_1^\tau\rangle_\tau dk\\ &=c_{-\mu,\tau}\,\mathrm{th}^p t\,\mathrm{ch}^{-\ell} tF\Big(\frac{\ell+p+q}{2},\frac{\ell+p-q+1-m_{2\alpha}}{2},\\ &\qquad p+\frac{m_\alpha+m_{2\alpha}+1}{2},\mathrm{th}^2 t\Big)\end{aligned}\tag{2.3}$$

where $\ell=i\mu+\rho$ and $c_{-\mu,\tau}$ is a constant. Setting $d_\alpha=-p(p+m_\alpha+m_{2\alpha}-1)+q(q+m_{2\alpha}-1)$ and $d_{2\alpha}=-4q(q+m_{2\alpha}-1)$ the function $\psi=\psi_{-\mu,\tau}$ satisfies

$$\begin{aligned}&\psi_{tt}+(m_\alpha+m_{2\alpha})\coth t\psi_t+m_{2\alpha}\,\mathrm{th}\,t\psi_t\\ &\quad=[d_\alpha\,\mathrm{sh}^{-2}t+d_{2\alpha}\,\mathrm{sh}^{-2}2t+\rho^2+\mu^2]\psi=0.\end{aligned}\tag{2.4}$$

3. We consider first the case $m_{2\alpha}=0$ and $m_\alpha=m$. These situations involve the Lobačevskij spaces (e.g. $G=SO_0(3,1)$ and $K=SO(3)$ with $m=2$) and the standard case of $G=SL(2,R)$ and $K=SO(2)$ with $m=1$. Resolvants were found in [2]–[6], [11] by different methods and expressed in terms of associated Legendre function or hypergeometric functions of other arguments. The results can easily be put into the present format as follows. We have $\rho=m/2$, $\ell=i\mu+m/2$, $d_\alpha=-p(p+m-1)$, and $d_{2\alpha}=0$ while $\tau\sim(p,0)$.

Theorem 2. *Resolvants for the case $m_\alpha=m$ and $m_{2\alpha}=0$ are given by*

$$\begin{aligned}\hat R^p(t,\mu)&=c_{-\mu,\tau}^{-1}\,\mathrm{sh}^{-p}\,t\psi_{-\mu,\tau}(a_tK)\\ &=\mathrm{ch}^{-p-\ell_t}F\Big(\frac{\ell+p}{2},\frac{\ell+p+1}{2},p+\frac{m+1}{2},\mathrm{th}^2 t\Big)\\ &=\frac{\Gamma(p+m/2+1/2)2^{p+m/2-1/2}}{\mathrm{sh}^{p+m/2-1/2}\,t}P_{i\mu-1/2}^{-p-m/2-1/2}(\mathrm{ch}\,t).\end{aligned}\tag{3.1}$$

These satisfy the resolvant initial conditions as well as the differential equations and splitting recursion relations below.

$$\hat R_{tt}^p+(2p+m)\coth t\hat R_t^p+\Big[p(p+m)+\mu^2+\Big(\frac{m}{2}\Big)^2\Big]\hat R^p=0\tag{3.2}$$

$$\hat R_t^p=\frac{-\mathrm{sh}\,t}{2p+m+1}\Big[p(p+m)+\mu^2+\Big(\frac{m}{2}\Big)^2\Big]\hat R^{p+1}\tag{3.3}$$

$$\hat R_t^p+(2p+m-1)\coth t\hat R^p=(2p+m-1)\,\mathrm{csch}\,t\hat R^{p-1}\tag{3.4}$$

The recursion relations can be found group theoretically by considering a full set of basis elements in the $\mathscr{H}_\mu$ spaces or simply by known recursion formulas for the associated Legendre functions. Their composition, with suitable index changes, yields (3.2) and this is what we mean by splitting (3.2). We remark in passing that resolvants are not unique since if we multiply $\hat{R}^m$ by a function $\varphi_m \in C^2$ such that $\varphi_m(0)=1$, $\varphi'_m(0)=0$, and $\varphi_0 \equiv 1$ for example then we would simply obtain different equations (3.2)–(3.4) while the resolvant initial conditions are preserved and for $m=0$ there arises again the $\hat{R}^0$ of (2.2).

4. In the case when $m_{2\alpha}=1$ we take $m_\alpha=m$ so that $d_{2\alpha}=-4q^2$ and $d_\alpha=-p(p+m)+q^2$ with $\rho=\dfrac{m}{2}+1$. We choose resolvants again in the form

$$\begin{aligned}\hat{R}^{p,q}(t,\mu)&=c^{-1}_{-\mu,\tau}\,\mathrm{sh}^{-p}\,t\psi_{-\mu,\tau}(a_tK)\\ &=\mathrm{ch}^{-p-2x}\,F\left(x+\frac{p+q}{2},\,x+\frac{p-q}{2},\,y,\,\mathrm{th}^2\,t\right)\end{aligned} \tag{4.1}$$

where $x=(1/2)\left(i\mu+\dfrac{m}{2}+1\right)=\ell/2$ and $y=p+\dfrac{m}{2}+1$. Using (2.4) one obtains

$$\begin{aligned}&\hat{R}^{p,q}_{tt}+[(2p+m+1)\coth t+\mathrm{th}\,t]\hat{R}^{p,q}_t\\ &\quad+\left[p(p+m+2)+\mu^2+\left(\frac{m}{2}+1\right)^2+q^2\,\mathrm{sech}^2\,t\right]\hat{R}^{p,q}=0.\end{aligned} \tag{4.2}$$

Theorem 3. *Resolvants for the case $m_\alpha=m$ and $m_{2\alpha}=1$ are given by (4.1) and satisfy (4.2) along with the resolvant initial conditions. There are various splitting recursion relations according as p or q change by 2 or (p,q) by $(\pm 1,\pm 1)$. We list these in the form*

$$\begin{aligned}\hat{R}^{p,q}_t=&\left[\frac{2\left(x+\frac{p+q}{2}\right)\left(x+\frac{p-q}{2}\right)}{y}-p-2x\right]\mathrm{th}\,t\hat{R}^{p,q}\\ &+\frac{2\left(x+\frac{p+q}{2}\right)\left(x+\frac{p-q}{2}\right)\left(y-x-\frac{p+q}{2}\right)\left(x+\frac{p-q}{2}-y\right)}{y^2(y+1)}\\ &\times\mathrm{sh}^2\,t\,\mathrm{th}\,t\hat{R}^{p+2,q}\end{aligned} \tag{4.3}$$

$$\begin{aligned}\hat{R}^{p,q}_t=&\,2(y-1)\coth t\,\mathrm{sech}^2\,t\hat{R}^{p-2,q}\\ &+\left[2(1-y)\coth t+\left\{\frac{2\left(x+\left(\frac{p+q}{2}\right)-1\right)\left(y-x-\frac{p-q}{2}-1\right)}{y-2}-q\right\}\mathrm{th}\,t\right]\hat{R}^{p,q}.\end{aligned} \tag{4.4}$$

$$\hat{R}^{p,q}_t=q\,\mathrm{th}\,t\hat{R}^{p,q}+\frac{2}{y}\left(x+\frac{p+q}{2}\right)\left(x+\frac{p-q}{2}-y\right)\mathrm{sh}\,t\hat{R}^{p+1,q+1} \tag{4.5}$$

(4.6) $\hat{R}^{p,q}_t = -q \,\mathrm{th}\, t\hat{R}^{p,q} - 2(y-1)\coth t\hat{R}^{p,q} + 2(y-1)\,\mathrm{csch}\, t\hat{R}^{p-1,q-1}$

(4.7) $\hat{R}^{p,q}_t = -q \,\mathrm{th}\, t\hat{R}^{p,q} - \dfrac{2}{y}\left(x + \dfrac{p-q}{2}\right)\left(y - x - \dfrac{p+q}{2}\right) \mathrm{sh}\, t\hat{R}^{p+1,q-1}$

(4.8) $\hat{R}^{p,q}_t = q \,\mathrm{th}\, t\hat{R}^{p,q} - 2(y-1)\coth t\hat{R}^{p,q} + 2(y-1)\,\mathrm{csch}\, t\hat{R}^{p-1,q+1}$

(4.9) $$\hat{R}^{p,q}_t = q \,\mathrm{th}\, t\hat{R}^{p,q} + \frac{2\coth t}{q+1}\left(x + \frac{p+q}{2}\right)\left(x + \frac{p-q}{2} - y\right)(\hat{R}^{p,q} - \hat{R}^{p,q+2})$$

(4.10) $$\hat{R}^{p,q}_t = -q \,\mathrm{th}\, t\hat{R}^{p,q} + \frac{2\coth t}{q-1}\left(x + \frac{p-q}{2}\right)\left(y - x - \frac{p+q}{2}\right)(\hat{R}^{p,q} - \hat{R}^{p,q-2}).$$

The recursion relations are obtained using the formula $d/dzF(a, b, c, z) = (ab/c)F(a+1, b+1, c+1, z)$ and various contiguity relations for hypergeometric functions. The cases $m_{2\alpha}=3$ or 7 can be treated in a similar manner. For the connection of the Fourier theory to the associated singular Cauchy problems see also [1]–[6], [11].

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