175. *Global Analytic-Hypoellipticity of the $\bar{\partial}$-Neumann Problem*

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1. Statement of Theorem. Let $M \subset \boldsymbol{C}^n$ be a domain with compact closure $\bar{M}$ and (real)-analytic boundary bM. We denote by r the distance function to bM measured as positive outside and negative inside M. We define Ω'_ρ as the tubular neighborhood of bM in $\boldsymbol{C}^n$ with small width ρ, and set $\Omega_\rho = \bar{M} \cap \Omega'_\rho$. By T_t we denote the subbundle of the complexified tangent bundle $\boldsymbol{C}T$ over Ω'_ρ of all vectors X with $\langle dr, X \rangle = 0$, where $\langle\ ,\ \rangle$ is the duality between covectors and vectors. Splitting $\boldsymbol{C}T$ as $\boldsymbol{C}T = T^{1,0} \oplus T^{0,1}$ with the subbundle $T^{1,0}$ of vectors of type $(1, 0)$ and its complex conjugate $T^{0,1}$, we set $T_t^{1,0} = T^{1,0} \cap T_t$ and $T_t^{0,1} = \overline{T_t^{1,0}}$. Then the *Levi form* at $\mathrm{P} \in \Omega'_\rho$ is defined on the fibre $(T_t^{1,0})_\mathrm{P}$ of $T_t^{1,0}$ at P by

$$(T_t^{1,0})_\mathrm{P} \times (T_t^{1,0})_\mathrm{P} \ni (X_1, X_2) \mapsto \langle \partial\bar{\partial} r, X_1 \wedge \bar{X}_2 \rangle.$$

Denote by $\mathcal{A}^{p,q}$ the space of forms of type (p, q) on $\bar{M}$ which have C^∞ extensions to $\boldsymbol{C}^n$, and define the L^2-inner product by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle dV, \qquad \varphi, \psi \in \mathcal{A}^{p,q},$$

with the pointwise inner product $\langle\ ,\ \rangle$ and the volume form dV on M. For the Cauchy-Riemann operator $\bar{\partial}: \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q}$ and its formal adjoint $\vartheta: \mathcal{A}^{p,q} \to \mathcal{A}^{p,q-1}$, integration by parts gives us

$$(\vartheta\varphi, \psi) = (\varphi, \bar{\partial}\psi) + \int_{bM} \langle \sigma(\vartheta, dr)\varphi, \psi \rangle dS,$$

where $\sigma(\cdot, dr)$ denotes the principal symbol of $\cdot$ at dr, and dS the volume form on bM. We set $\mathcal{D}^{p,q} = \{\varphi \in \mathcal{A}^{p,q}; \sigma(\vartheta, dr)\varphi = 0 \text{ on } bM\}$, and define a quadratic form on $\mathcal{D}^{p,q}$ by

$$Q(\varphi, \psi) = (\bar{\partial}\varphi, \bar{\partial}\psi) + (\vartheta\varphi, \vartheta\psi) + (\varphi, \psi), \qquad \varphi, \psi \in \mathcal{D}^{p,q}.$$

Consider the following variational problem (cf. [1], [3]): Given $\lambda \in \boldsymbol{C}$ and $\alpha \in \mathcal{A}^{p,q}$ with $q > 0$, find $\varphi \in \mathcal{D}^{p,q}$ such that

$$(1) \qquad Q(\varphi, \psi) + (\lambda\varphi, \psi) = (\alpha, \psi) \qquad \text{for all } \psi \in \mathcal{D}^{p,q}.$$

Now we have

Theorem. *If the Levi form is non-degenerate and does not have exactly q negative eigenvalues in Ω'_ρ, then every solution φ of the equation* (1) *is analytic in Ω_ρ whenever α is analytic there.*

We remark that this Theorem can easily be generalized to the case of domains M in complex manifolds with analytic hermitian metric.

2. A priori estimate and a special vector field. Letting $\mathcal{A}_\rho^{p,q}$ denote the space of elements in $\mathcal{A}^{p,q}$ supported in Ω_ρ and setting $\mathcal{D}_\rho^{p,q} = \mathcal{A}_\rho^{p,q} \cap \mathcal{D}^{p,q}$, we define an operator $\overline{n}: \mathcal{A}_\rho^{p,q} \to \mathcal{A}_\rho^{p,q}$ of order zero by $\overline{n} = \sigma(-\bar{\partial}\vartheta, dr)$, which is an orthogonal projection relative to the inner product $\langle\ ,\ \rangle$. Denoting by $\Gamma(\cdot)$ the space of sections of $\cdot$ over Ω'_ρ and letting $\nabla_X: \mathcal{A}_\rho^{p,q} \to \mathcal{A}_\rho^{p,q}$ be the (complex) covariant differentiation along $X \in \Gamma(CT)$, we set $\tilde{\nabla}_X = \overline{n}\nabla_X\overline{n} + (1-\overline{n})\nabla_X(1-\overline{n})$, which maps $\mathcal{D}_\rho^{p,q}$ into itself if $X \in \Gamma(T_t)$.

By means of local orthonormal basis $(L_1, \cdots, L_n)$ of $T^{1,0}$ with $L_n = R =$ the dual of ∂r, we define a norm on $\mathcal{D}_\rho^{p,q}$ by

$$N(\varphi) = \left(\int_M \left(\sum_{i=1}^{n} |\tilde{\nabla}_{\bar{L}_i}\varphi|^2 + \sum_{i=1}^{n-1} |\tilde{\nabla}_{L_i}\varphi|^2 + |\varphi|^2\right) dV\right)^{1/2}$$

with $|\varphi|^2 = \langle \varphi, \varphi \rangle$, which is well-defined when ρ is small.

We say the *basic estimate* holds in $\mathcal{D}^{p,q}$ if for some $C>0$,

$$\int_{bM} |\varphi|^2 dS \leqslant CQ(\varphi, \varphi) \qquad \text{for all } \varphi \in \mathcal{D}^{p,q},$$

an estimate guaranteed by our assumption (see [2]). Now one has

Lemma 1 (a priori estimate). *If the basic estimate holds in $\mathcal{D}^{p,q}$, then there exists a constant $C>0$ such that*

$$C^{-1}N(\varphi)^2 \leqslant Q(\varphi, \varphi) \leqslant CN(\varphi)^2 \qquad \textit{for all } \varphi \in \mathcal{D}_\rho^{p,q}.$$

We need commutator estimates as usual in proving the regularity. The vector field Y given in the following lemma will play an essential role in our commutator estimates.

Lemma 2. *Suppose that the Levi form is non-degenerate in Ω'_ρ. If ρ is sufficiently small, then there exists an analytic vector field $Y \in \Gamma(T_t)$ with $\overline{Y} = -Y$ such that*

$$\langle \partial r, [X, Y] \rangle = 0 \textit{ in } \Omega'_\rho \qquad \textit{for all } X \in \Gamma(T_t^{1,0} \oplus T_t^{0,1}),$$
$$\langle \partial r, [\bar{R}, Y] \rangle = 0 \textit{ on } bM \quad \textit{and} \quad \langle \partial r, Y \rangle = 1 \textit{ on } bM,$$

where $[X_1, X_2]$ denotes the commutator $X_1X_2 - X_2X_1$.

Taking ample vector fields $Z_1, \cdots, Z_{2n} \in \Gamma(T_t^{1,0} \oplus T_t^{0,1})$ which are analytic, and letting $|K| = l$ and $\tilde{\nabla}_Z^K = \tilde{\nabla}_{Z_{\kappa_1}} \cdots \tilde{\nabla}_{Z_{\kappa_l}}$ for ordered integers $K = (\kappa_1, \cdots, \kappa_l)$ with $1 \leqslant \kappa_i \leqslant 2n$, we set

$$N(\varphi; l, m) = (l+m)!^{-1} \max_{|K|=l} N(\tilde{\nabla}_Z^K \tilde{\nabla}_Y^m \varphi) \qquad \text{for } \varphi \in \mathcal{D}_\rho^{p,q}.$$

3. Sketch of the proof of Theorem. Recall that the solution φ of the equation (1) satisfies the second order differential equation

$$\Box\varphi + (\lambda+1)\varphi = \alpha, \qquad \text{where } \Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}.$$

Then, if we notice that the operator $\Box$ has analytic coefficients and is of elliptic type, the analyticity of the solution φ near bM will follow by virtue of the Holmgren's theorem from that of the Cauchy data of φ on bM (see [4]), a fact which is equivalent to

$$(2) \qquad N(\zeta\varphi; l, m) \leqslant C_0 C_1^l C_2^m \qquad \text{for all } l \geqslant 0 \text{ and } m \geqslant 0,$$

where $\zeta = \zeta(r)$ is a function of r supported in Ω'_ρ and $\zeta = 1$ near bM. In view of Lemma 1, the inequalities (2) will be obtained by estimating the commutators

(3) $Q(D\zeta\varphi, D\zeta\varphi)-Q(\zeta\varphi, D^*D\zeta\varphi), \qquad D=\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m,\ |K|=l,$

with D^* denoting the formal adjoint of D, for $\zeta\varphi$ satisfies the $\bar\partial$-*Neumann conditions* $\zeta\varphi\in\mathcal{D}^{p,q}$ and $\bar\partial(\zeta\varphi)\in\mathcal{D}^{p,q+1}$ so that

$$Q(\zeta\varphi, D^*D\zeta\varphi)+(\lambda\zeta\varphi, D^*D\zeta\varphi)=((\Box+\lambda+1)\zeta\varphi, D^*D\zeta\varphi).$$

The estimates of (3) are carried out by using the properties of Y in Lemma 2, and the proof of (2) can be done in two step induction. One proves first the inequalities (2) for $l=0$ by inductive use of the corresponding estimates of (3). The case $l>0$ can be obtained by combining the estimates of (3) with the results for $l=0$.

References

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