28. *On a Nonlinear Noncontractive Semigroup*

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(Comm. by Kôsaku YOSIDA, M. J. A., March 12, 1976)

1. Introduction and Theorem. Let X be a Banach space with norm $\|\cdot\|$. We consider an operator $A: D(A)\subset X\to X$ such that i) $D(A)\ni 0$, $A0=0$ ii) $R(I+\lambda A)=X$ *for all* $\lambda>0$ iii) *there exists a constant* $M>0$ *such that for all* $\lambda>0$ *and* $x, y\in X$,

$$\|(I+\lambda A)^{-1}x-(I+\lambda A)^{-1}y\|\leq M\,\|x-y\|.$$

Let $J_\lambda=(I+\lambda A)^{-1}$ be Fréchet differentiable at every $x\in X$. Then $F(\lambda)=J'_\lambda[x+\lambda Ax]\in B(X,X)(x\in D(A))$ satisfies the first resolvent equation; $\lambda F(\lambda)-\mu F(\mu)=(\lambda-\mu)F(\mu)F(\lambda)$ (see [3] or [4]). Hence it follows that there exists a linear operator $A'[x]: D(A'[x])\to X$ such that $F(\lambda)=(I+\lambda A'[x])^{-1}$. Such an operator A is said to be R-defferentiable and $A'[x]$ the R-derivative of A at $x\in D(A)$.

The notion of R-differentiable operators was introduced by M. Iannelli to construct nonlinear noncontractive semigroups. In this note, we shall consider an R-differentiable opetator A such that $A'[x]$ satisfies a hyperbolic-type condition. We shall show that the infinitesimal generator of a semigroup associated with A, coincides with A on a subspace of X. Only the result and an outline of its proof are presented here and the details will be published elsewhere. Our result is following

Theorem. *Let A be an R-differentiable operator such that:*

(I) $A'[x]$ *is a closed linear operator for all* $x\in D(A)$,

(II) *there exists a Banach space Y which is densely and continuously embedded in X,*

(S_1) *for any finite family* $\{x_1,\cdots,x_n\}\subset D(A)$,

$$\left\|\prod_{i=1}^{n}(I+\lambda A'[x_i])^{-1}\right\|_X\leq M,$$

(S_2) $(I+\lambda A'[x])^{-1}(Y)\subset Y$ *for each* $x\in D(A)$, *and for* $\{x_i\}$ *stated in* (S_1),

$$\left\|\prod_{i=1}^{n}(I+\lambda A'[x_i])^{-1}\right\|_Y\leq K_1,$$

(III) $Y\subset D(A)$, $Y\subset D(A'[x])$ *for each* $x\in D(A)$, *and*

$$\|A'[x]-A'[y]\|_{Y,X}\leq K_2\,\|x-y\|.$$

Here $K_i, i=1,2$ *are constants and* $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_{Y,X}$ *denote the norms in* $B(X,X), B(Y,Y), B(Y,X)$ *respectively.*

Then there exists a unique semigroup $\{G(t)\}_{t\geq 0}$ *such that*

(a) $G(t)x=\lim_{n\to\infty}(I+(t/n)A)^{-n}x$ *for all* $t\geq 0$ *and* $x\in X$,

(b) $\|G(t)x-G(t)y\| \leq M\|x-y\|$ *for all* $x, y \in X$,
(c) $G(t)G(s)=G(t+s)$, $G(0)=I$,
(d) $G(t)$ *is strongly continuous in* t,
(e) $D_t^+ G(t)y|_{t=0}=-Ay$ *for all* $y \in Y$.
Here D^+ *denotes the right derivative in the strong topology of* X.

It is shown in [4] that (a)～(d) of our theorem are consequences of only (S_1). To prove (e), we need some lemmas as [3] to represent $G(t)$ in an integral form involving a one parameter family of linear operators. Almost all of our assumptions on $A'[x]$ are similar to those of T.Kato [5]. The assumption "$Y \subset D(A)$" in (III) may be seen unnatural, but we have the following

Proposition 1. *Let the assumptions of the theorem be satisfied except* "$Y \subset D(A)$". *Then* $D(A)$ *is dense in* X.

2. Some lemmas. In the following, let all assumptions of the theorem be always satisfied. Let $C(T)=C([0,T]\times[0,1]:X)$ be the space of continuous functions from $[0,T]\times[0,1]$ to X. For any $u \in C(T)$ and any zero sequence $\{\lambda_n\}$ there exists an approximate sequence $\{u_n\}$ such that

$$(2.1)\quad \begin{aligned} &u_n(t,\sigma)=u_n(i\lambda_n, j\lambda_n) \in D(A) \qquad \text{if } i\lambda_n \leq t<(i+1)\lambda_n \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \qquad j\lambda_n \leq \sigma <(j+1)\lambda_n, \\ &\lim_{n\to\infty} \sup_{(t,\sigma)\in[0,T]\times[0,1]} \|u_n(t,\sigma)-u(t,\sigma)\|=0. \end{aligned}$$

Lemma 2. *Let* $u \in C(T)$ *and* $\{u_n\}$ *be an approximate sequence for* u. *Then there exists*

$$U\{u,\sigma\}(t,0)x=\lim_{\substack{n\to\infty \\ n\lambda_n \to t}} \prod_{i=1}^{n} (I+\lambda_n A'[u_n(i\lambda_n,\sigma)])^{-1}x \qquad \textit{for all } x \in X.$$

Moreover, for $u, v \in C(T)$ *and* $y \in Y$, *we have*

$$(2.2)\quad \begin{aligned} &\sup_{(t,\sigma)\in[0,T]\times[0,1]} \|U\{u,\sigma\}(t,0)y-U\{v,\sigma\}(t,0)y\| \\ &\qquad \leq K_1K_2MT\|y\|_Y \sup_{(t,\sigma)\in[0,T]\times[0,1]} \|u(t,\sigma)-v(t,\sigma)\|. \end{aligned}$$

In particular, from this estimate, we see that $U\{u,\sigma\}$ *is defined independently of the choice of the approximate sequence* $\{u_n\}$.

For the proof, we have for $y \in Y$ and $m \leq n$

$$\begin{aligned} &\prod_{i=1}^{m} (I+\lambda A'[u_m(i\lambda,\sigma)])^{-1}y-\prod_{i=1}^{n} (I+\mu A'[u_n(i\mu,\sigma)])^{-1}y \\ &\quad =\sum_{i=1}^{m-1} \beta^{n-i}\alpha^i \Bigg(\sum_{(m-i,0)}^{(m,n)} \prod_{p=1}^{n} (I+\mu A'[u_n(c_p\lambda,\sigma)])^{-1}\Bigg) \prod_{i=1}^{m-i} (I+\lambda A'[u_m(i\lambda,\sigma)])^{-1}y \\ &\qquad +\sum_{i=m}^{n} \alpha^m\beta^{i-m}\Bigg(\sum_{(1,n-i+1)}^{(m,n)} \prod_{p=1}^{i-1} (I+\mu A'[u_n(c_p\lambda,\sigma)])^{-1}\Bigg) (I+\mu A'[u_n(\lambda,\sigma)])^{-1} \\ &\qquad \times \prod_{i=1}^{n-i} (I+\mu A'[u_n(i\mu,\sigma)])^{-1}y \\ &\qquad +\mu \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta^{j-i}\alpha^i \Bigg(\sum_{(m-i,n-j)}^{(m,n)} \prod_{p=1}^{j} (I+\mu A'[u_n(c_p\lambda,\sigma)])^{-1}\Bigg) \end{aligned}$$

$$\times(I+\mu A'[u_n((m-i)\lambda,\sigma)])^{-1}\{A'[u_n((n-j)\mu,\sigma)] - A'[u_m((m-i)\lambda,\sigma)]\} \times\prod_{k=1}^{n-j}(I+\mu A'[u_n(k\mu,\sigma)])^{-1}y.$$

Here $j\wedge i=\min\{j,i\}$, $\alpha=\mu/\lambda$, $\alpha+\beta=1$. $\sum_{(i,j)}^{(m,n)}$ is interpreted as follows: For any lattice point (k,l) $(k\geq 1, l\geq 1)$, we choose as admissible k-segments two line segments joining it to $(k-1,l-1)$ or to $(k,l-1)$. Then $\sum_{(i,j)}^{(m,n)}$ runs over all of $\{c_p\}$, each $\{c_p\}$ denoting the shortest path of admissible segments from (m,n) to (i,j). Thus $\sum_{(i,j)}^{(m,n)}$ contains $\binom{n-j}{m-i}$ terms in it. This formula is essentially due to [2]. In [2] it is obtained in a form of norm inequality for the case that $(I+\lambda A'[x])^{-1}$ is a contraction mapping. In our case, we use the linearity of operators to have the equality. Then the same method of [2] is applicable to prove Lemma 2.

Lemma 3. *For any $u\in C(T)$, $U\{u,\sigma\}(t,0)x$ which has been defined in Lemma 2, belongs to $C(T)$ for each $x\in X$.*

Definition 4. *Let $u\in C(T)$ and $\{u_n\}$ be an approximate sequence for u. We define for $(t,s)\in[0,T]\times[0,1]$*

$$(G\{T,x\}u)(t,s)=\int_0^s U\{u,\sigma\}(t,0)x\,d\sigma,$$

$$(G_n\{T,x\}u)(t,s)=\int_0^s\prod_{i=1}^{n}(I+\lambda_nA'[u_n(i\lambda_n,\sigma)])^{-1}x\,d\sigma.$$

By Lemma 3, $G\{T,x\}$ maps $C(T)$ into itself.

Lemma 5. *We have*

$$\lim_{n\to\infty}\sup_{(t,\sigma)\in[0,T]\times[0,1]}\|(G\{T,x\}u)(t,s)-(G_n\{T,x\}u)(t,s)\|=0.$$

Lemma 6. *Let $T>0$ be an arbitrary fixed number and, for $(t,s)\in[0,T]\times[0,1]$ and $y\in Y$, set $u(t,s)=\lim_{\substack{n\to\infty\\ n\lambda_n\to t}}(I+\lambda_nA)^{-n}sy$. $u(t,s)$ exists and belongs to $C(T)$ by Theorem 3.1 of [4]. Then we have*

$$(G\{T,y\}u)(t,s)=u(t,s).$$

For the proof, we notice that $g_n(t,s)=(G_n\{T,y\}g_n)(t,s)$, where $g_n(t,\sigma)=(I+\lambda_nA)^{-i}\sigma y$ for $i\lambda_n\leq t<(i+1)\lambda_n$ and $0\leq\sigma\leq 1$.

3. Sketch of the proof of (e).

First we have

$$\lim_{\lambda\to 0}(I+\lambda A'[J_\lambda\sigma y])^{-1}x=x \qquad \textit{for all } x\in X$$

and, using this relation, we have

$$Ay=\int_0^1 A'[\sigma y]y\,d\sigma \qquad \textit{for } y\in Y.$$

On the other hand, we notice that $D_t^+U\{u,\sigma\}(t,0)|_{t=0}=-A'[\sigma y]y$. Then by Lemma 6 and facts stated above, we get

$$\lim_{t\downarrow 0}(G(t)y-y)/t=\int_0^1 -A'[\sigma y]y\,d\sigma=-Ay.$$

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