25. *Putcha's Problem on Maximal Cancellative Subsemigroups*

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(Comm. by Kenjiro SHODA, M. J. A., March 12, 1976)

1. Introduction. Let S be a commutative archimedean semigroup without idempotent ([1], [3], [5]). M. S. Putcha asked the following question in his recent paper [4].

Is every maximal cancellative subsemigroup of S necessarily archimedean?

In this paper the author negatively answers this question by exhibiting a counter example and discusses a further problem. Throughout this paper, Z denotes the set of integers, Z_+ the set of positive integers and Z_+^0 the set of nonnegative integers. Let S be a commutative semigroup and let a be any element of S. Define ρ_a on S by

$$x \rho_a y \quad \text{if and only if} \quad a^m x = a^n y \quad \text{for some} \quad m, n \in Z_+.$$

Then ρ_a is a congruence on S, and if S is a commutative archimedean semigroup without idempotent, then S/ρ_a is a group [5], [6]. Let $G_a = S/\rho_a$. G_a is called the *structure group* of S with respect to a. A commutative semigroup S is called power joined if, for any $a, b \in S$, there are $m, n \in Z_+$ such that $a^m = b^n$.

Putcha's question is affirmative if G_a is torsion. It is more strongly stated as follows:

Proposition 1.1. *Let S be a commutative archimedean semigroup without idempotent. If a structure group of S is torsion, then every subsemigroup of S is archimedean.*

Proof. According to [2], S is power joined if and only if G_a is torsion for some $a \in S$, equivalently for all $a \in S$. Every subsemigroup of S is power joined, hence archimedean.

Accordingly Putcha's question is interesting only in the case G_a is not torsion.

2. Counter example. Let G be the free abelian group of rank $r \geqq 2$, where r may be infinite, but we assume $2 \leqq r \leqq \aleph_0$ for our convenience. However this restriction will be easily removed later. Every element λ of G will be expressed by

$$\lambda = (\lambda_1, \cdots, \lambda_i, \cdots) \quad \text{or} \quad (\lambda_i)$$

where $\lambda_i \in Z$ for all $i \in Z_+$, but if $r = \aleph_0$, only a finite number of λ_i's are not zero. The operation is defined by $(\lambda_i) + (\mu_i) = (\lambda_i + \mu_i)$ and the identity is $\mathbf{0} = (0)$. Define subsemigroups H and E of G by

$$H=\{\lambda \in G: \lambda_i \geqq 0 \quad \text{for all } i \in Z_+\},$$
$$E=\{\lambda \in H: \lambda_{2i+1}=0 \quad \text{for all } i \in Z_+^0\}.$$

For each $\lambda=(\lambda_i)$ of G, we define $\|\lambda\|$ by

$$\|\lambda\|=\sum_{\lambda_i\neq 0} \lambda_i, \qquad \|\mathbf{0}\|=0.$$

Let $S=H\cup(G\times Z_+^0)$ be the set union of the set H and the product set $G\times Z_+^0$. Elements of H are denoted by $\lambda, \mu, \cdots$; those of $G\times Z_+^0$ are denoted by $(\lambda, x), (\mu, y), \cdots$ where $\lambda, \mu \in G, x, y \in Z_+^0$. Define the commutative binary operation in S as follows:

$$\lambda\cdot\mu=\begin{cases}(\lambda+\mu, 3) & \lambda, \mu \in H\backslash E.\\ (\lambda+\mu, \|\mu\|+2) & \lambda \in H\backslash E, \mu \in E.\\ (\lambda+\mu, \|\lambda+\mu\|+1) & \lambda, \mu \in E.\end{cases}$$

$$\lambda\cdot(\mu, x)=\begin{cases}(\lambda+\mu, x+2) & \lambda \in H\backslash E, \mu \in G.\\ (\lambda+\mu, x+\|\lambda\|+1) & \lambda \in E, \mu \in G.\end{cases}$$

$$(\lambda, x)\cdot(\mu, y)= (\lambda+\mu, x+y+1) \qquad \lambda, \mu \in G.$$

The subsemigroup $G\times Z_+^0$ of S is isomorphic to the direct product of G and Z_+ under addition, and hence $G\times Z_+$ is archimedean. Furthermore S is an inflation [1] of $G\times Z_+^0$ determined by the map $\varphi: H\to G\times Z_+^0$ where φ is defined by

$$\varphi(\lambda)=\begin{cases}(\lambda, 1) & \text{if } \lambda \in H\backslash E\\ (\lambda, \|\lambda\|) & \text{if } \lambda \in E.\end{cases}$$

Therefore S is a commutative archimedean semigroup. Since $G\times Z_+^0$ has no idempotent, S has no idempotent.

Let

$$T=H\cup\{(\lambda, x): \lambda \in H\backslash E, x\geqq 2\}\cup\{(\lambda, \|\lambda\|+x): \lambda \in E, x\geqq 1\}.$$

From the definition of multiplication in S, we see that T is a subsemigroup of S. Let $L=T\backslash H$. L is a cancellative ideal of T. It is easily seen that $\lambda\cdot(\nu, x)=\mu\cdot(\nu, x)$ implies $\lambda=\mu$. The other cases of cancellation of T is shown by cancellation of L. Therefore T is cancellative.

Let $\theta: S\to G$ be the homomorphism defined by

$$\theta(\lambda)=\lambda \qquad \text{if } \lambda \in H$$
$$\theta(\lambda, x)=\lambda \qquad \text{if } (\lambda, x) \in G\times Z_+^0.$$

Note that θ is nothing but $S\to G_0=S/\rho_0$.

Let M be a cancellative subsemigroup of S properly containing T. Suppose M is archimedean. Then $\theta(M)$ is archimedean. Since the subsemigroup H of G contains the identity $\mathbf{0}$ of G, $\theta(H)$ contains $\mathbf{0}$, and hence $\theta(M)$ contains $\mathbf{0}$. It is, therefore, a subgroup of G which contains the subsemigroup H of G. But $G=\theta(M)$ since G is generated by H. Consider $\lambda \in G$ defined by

$$\lambda_i=\begin{cases}-1 & i=1\\ \ \ 0 & i\neq 1.\end{cases}$$

Then $(\lambda, x) \in M$ for some $x \in Z_+^0$. Choose $\nu \in E$ such that $\|\nu\|=x+2$. Let $\mu=-\lambda+\nu$. Then $\mu \in H\backslash E$, so $\mu \in T$ and

$$\mu\cdot(\lambda, x)=(\lambda+\mu, x+2)=(\nu, \|\nu\|),$$

so that, $(\nu, \|\nu\|)\in M$. On the other hand,

$$\nu\cdot(\nu, \|\nu\|)=(2\nu, 2\|\nu\|+1)=(2\nu, \|2\nu\|+1)=\nu\cdot\nu$$

but $(\nu, \|\nu\|)\neq\nu$. This contradicts cancellation of M. Hence no cancellative subsemigroup which properly contains T is archimedean.

We can remove the restriction "$\leqq\aleph_0$". Let G_1 be the free abelian group of rank $>\aleph_0$. The above G is regarded as a subgroup of G_1. Let H and E be the subsemigroups of G defined as before and let $S_1=H\cup(G_1\times Z_+^0)$; the operation in S_1 is defined in the same way as in S except replacing G by G_1. T is exactly the same as before and $\theta_1: S_1\to G_1$ is similarly defined as θ. If M_1 is a cancellative subsemigroup of S_1 and if $T\subsetneqq M_1\subset S$, then $G_1=\theta_1(M_1)$ and we have the same conclusion.

3. **Remark.** Let D be a commutative semigroup. If there is an element a of D such that, for every $b\in D, a^m=bc$ for some $c\in D$ and some $m\in Z_+$, then D is called subarchimedean. Let G be the free abelian group of rank $r, 2\leqq r\leqq\aleph_0$. In this section, we note that the T in Section 2 is contained in a subarchimedean maximal cancellative subsemigroup M_0 of S. Let $A=\{\lambda\in G: \lambda_{2i}=0 \text{ for all } i\in Z_+\}, F=\{\lambda\in A: \|\lambda\|\geqq 0\}$.

(3.1) *Let X be a subsemigroup of A such that $F\subsetneqq X\subset A$. Then X contains an element $\nu\in G$ such that $\nu\neq 0$ and $\nu_{2i+1}\leqq 0$ for all $i\in Z_+^0$.*

As the dual of A, we define $B=\{\lambda\in G: \lambda_{2i+1}=0 \text{ for all } i\in Z_+^0\}$. Then G is the direct sum of A and B: $G=A+B$. Let $\lambda\in G$. The projections of λ into A and B are denoted by λ_A and λ_B respectively: $\lambda=\lambda_A+\lambda_B$. Now define $\tilde{H}=\{\lambda\in G: \lambda_A\in F\}$. E and H were defined in Section 2 and $\bar{E}$ denotes the subgroup of G generated by E. Then $H\subset\tilde{H}$ and $\tilde{H}=F+B$.

Let

$$F_0=\{\lambda\in F: \|\lambda\|=0\}, \qquad F_+=\{\lambda\in F: \|\lambda\|>0\},$$
$$\tilde{H}_0=\{\lambda\in G: \lambda_A\in F_0\}, \qquad \tilde{H}_+=\{\lambda\in G: \lambda_A\in F_+\}.$$

F_0 is a subgroup of F, and F_+ is an ideal of F; $H\backslash E$ is an ideal of H; $\tilde{H}\backslash E$ is an ideal of $\tilde{H}$.

(3.2) *Let $\lambda, \mu\in\tilde{H}$. Then $\lambda+\mu\in\tilde{H}_0$ if and only if $\lambda, \mu\in\tilde{H}_0$.*

Further, consider the subsets Y of $\tilde{H}_+\backslash H$ satisfying that $\lambda+\mu\notin H$ for every distinct $\lambda, \mu\in Y$. Let C be a maximal such set Y. Such a Y exists. For example, choose $\lambda\in\tilde{H}_+\backslash H$ with $\lambda_2<0$, and then define $Y=\{m\lambda: m\in Z_+\}$. Existence of maximal one is due to Zorn's lemma.

Let $D=\tilde{H}_+\backslash(H\cup C)$, i.e., $\tilde{H}_+\backslash H=C\cup D$. Now define subsets of S as follows:

$$T_C=\{(\lambda, x): \lambda\in C, x\geqq 0\}, \qquad T_D=\{(\lambda, x): \lambda\in D, x\geqq 1\},$$
$$T_{F_0}=\{(\lambda, x): \lambda\in\tilde{H}_0\backslash H, x\geqq\|\lambda\|\}.$$

For our convenience the sets appearing in Section 2 are denoted by

$$T_1=\{(\lambda, x): \lambda \in H\backslash E, x \geqq 2\}, \qquad T_2=\{(\lambda, x): \lambda \in E, x \geqq \|\lambda\|+1\}.$$

Recall $T=H \cup T_1 \cup T_2$. Finally we define M_0 by

$$M_0=T \cup T_C \cup T_D \cup T_{F_0}.$$

Then we can show that M_0 is a maximal cancellative subsemigroup of S and M_0 is subarchimedean.

The S given in Sections 2 and 3 has also a maximal cancellative subsemigroup M_2 which is archimedean, and at the same time an ideal of S. $M_2=\{(\lambda, x): \lambda \in G, x \in Z_+^0\}$.

The following problems are raised.

Problem 1. Assume that S is a commutative archimedean semigroup without idempotent and a structure group of S is isomorphic to Z. Then is Putcha's question affirmative?

Problem 2. If S is a commutative archimedean semigroup without idempotent, is every maximal cancellative subsemigroup necessarily subarchimedean? Does there exist a maximal cancellative subsemigroup which is archimedean?

References

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