50. On an Explicit Formula for Class-1 "Whittaker Functions" on GL_n over $\mathfrak{P}$-adic Fields

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0. "Whittaker functions" on $\mathfrak{P}$-adic linear groups have been studied by several authors (see e.g. [2] and [3]). In this note, we present an explicit formula for the class-1 "Whittaker functions" on $GL_n(k)$, where k is a non archimedean local field.

1. Let k be a finite extension of the p-adic fied $\boldsymbol{Q}_p$ and let $\mathcal{O}$ be the ring of integers of k. Choose a generator π of the maximal ideal of $\mathcal{O}$ and denote by q the cardinality of the residue class field of k. Set $G=GL_n(k)$ and $K=GL_n(\mathcal{O})$. Then K is a maximal compact open subgroup of G. The invariant measure of G is normalized so that the total volume of K is equal to 1. Denote by $L_0(G,K)$ the space of complex valued compactly-supported bi-K-invariant functions on G. Then $L_0(G,K)$ is a commutative subalgebra of the group ring $L^1(G)$ of G. We denote by N the group of $n\times n$ upper triangular unipotent matrices with entries in k. Choose a character ψ of the additive group of k which is trivial on $\mathcal{O}$ but not trivial on $\pi^{-1}\mathcal{O}$. Denote by the same letter ψ the character of N given by $\psi(x)=\prod_{i=1}^{n-1}\psi(x_{ii+1})$, where x_{ii+1} is the $(i,i+1)$ entry of x.

For each algebra homomorphism λ of $L_0(G,K)$ into $\boldsymbol{C}$, it is known that there uniquely exists a function $W_\lambda(g)$ on G which satisfies the following conditions (1), (2) and (3).

$$W_\lambda(xg)=\psi(x)W_\lambda(g) \qquad (\forall x\in N), \tag{1}$$

$$\int_G W_\lambda(gx)\varphi(x)dx=\lambda(\varphi)W_\lambda(g) \qquad (\forall\varphi\in L_0(G,K)), \tag{2}$$

$$W_\lambda(1)=1. \tag{3}$$

The function W_λ is said to be the *class-*1 *"Whittaker function" on G associated with the homomorphism λ of $L_0(G,K)$ into* $\boldsymbol{C}$.

For each n-tuple $f=(f_1,f_2,\cdots,f_n)$ of integers, we denote by π^f the diagonal matrix whose i-th diagonal entry is π^{f_i} $(i=1,\cdots,n)$. Set $w_\lambda(f)=W_\lambda(\pi^f)$. It is known that $G=\bigcup_{f\in \boldsymbol{Z}^n} N\pi^f K$ (disjoint union). To evaluate W_λ on G, it is sufficient to know $w_\lambda(f)$ for all $f\in\boldsymbol{Z}^n$, since W_λ is right K-invariant and satisfies (1). Since the conductor of ψ is $\mathcal{O}$, it follows easily from (1) that $w_\lambda(f)$ is zero unless $f_1\geq f_2\geq\cdots\geq f_n$.

For $i=1,2,\cdots,n$, let φ_i be the characteristic function of the double

K-coset $K\pi^{f^i}K$, where $f^i=(\overbrace{1,1,\cdots,1}^{i},0,0,\cdots,0)$. It is known that $L_0(G,K)$ is isomorphic to the polynomial ring generated by $\varphi_1,\varphi_2,\cdots,\varphi_n$. Set $\lambda_i=\lambda(\varphi_i)$, $(i=1,2,\cdots,n)$ and choose n complex numbers $\mu_1,\mu_2,\cdots,\mu_n$ so that the i-th elementary symmetric function of μ_j's is equal to $q^{i(i-1)/2}\lambda_i$ $(i=1,2,\cdots,n)$. Let μ be the diagonal matrix whose i-th diagonal entry is μ_i for $i=1,2,\cdots,n$. Since $\lambda_n\neq 0$, $\mu\in GL_n(\boldsymbol{C})$.

For $f=(f_1,f_2,\cdots,f_n)\in \boldsymbol{Z}^n$, denote by χ_f the character of the irreducible representation of $GL_n(\boldsymbol{C})$ with the highest weight f, if $f_1\geq f_2\geq\cdots\geq f_n$. Unless $f_1\geq f_2\geq\cdots\geq f_n$, set $\chi_f=0$.

Theorem. *Notations and assumptions being as above, we have,*

$$W_\lambda(\pi^f)=q^{\sum_{i=1}^{n}(i-n)f_i}\chi_f(\mu)\qquad (f\in \boldsymbol{Z}^n),$$

where

$$(4)\qquad \chi_f(\mu)=\begin{cases}\dfrac{\begin{vmatrix}\mu_1^{f_1+n-1} & \mu_2^{f_1+n-1} & \cdots & \mu_n^{f_1+n-1}\\ \vdots & \vdots & & \vdots\\ \mu_1^{f_n} & \mu_2^{f_n} & \cdots & \mu_n^{f_n}\end{vmatrix}}{\prod_{i<j}(\mu_i-\mu_j)}, & \text{if } f_1\geq\cdots\geq f_n\\ 0, \text{ otherwise.}\end{cases}$$

Proof. We first prove the following sublemma:

Sublemma. (See Lemma 11 of [4].) *Set $N_{\mathcal{O}}=N\cap K$ and denote by I_i the set of all the n-tuples $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$ of non-negative integers which satisfy $\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n=i$. Further, set $N_{\mathcal{O}}(\varepsilon)=N_{\mathcal{O}}\cap\pi^\varepsilon K\pi^{-\varepsilon}$. Then we have*

$$(5)\qquad K\pi^{f^i}K=\bigcup_{\varepsilon\in I_i}\ \bigcup_{x\in N_{\mathcal{O}}/N_{\mathcal{O}}(\varepsilon)} x\pi^\varepsilon K\qquad (\textit{disjoint union}).$$

Proof. Set $e_i=(\underbrace{0,0,\cdots,0}_{n-i},\underbrace{1,1,\cdots,1}_{i})$. Denote by B the subgroup of K consisting of all matrices in K whose subdiagonal entries are all in $\pi\mathcal{O}$. It is known (see [1]) that $K=\bigcup_{w\in W}BwB$ (disjoint union), where W is the group of all permutation matrices in K. Since $(\pi^{e_i})^{-1}B(\pi^{e_i})\in K$ and $Bw\pi^{e_i}w^{-1}\subset\bigcup_{\varepsilon\in I_k}N_{\mathcal{O}}\pi^\varepsilon K$ ($\forall w\in W$), $K\pi^{f^i}K=K\pi^{e_i}K\subset\bigcup_{w\in W}Bw\pi^{e_i}K\subset\bigcup_{\varepsilon\in I_i}N_{\mathcal{O}}\pi^\varepsilon K$. Thus the left side of (5) is a subset of the right. Since the inverse inclusion relation is obvious, we obtain the sublemma.

It follows from the sublemma that if $f=(f_1,f_2,\cdots,f_n)\in\boldsymbol{Z}^n$ and $f_1\geq f_2\geq\cdots\geq f_n$,

$$\lambda_i w_\lambda(f)=\int_G W_\lambda(\pi^f x)\varphi_i(x)dx=\sum_{\varepsilon\in I_i}|N_{\mathcal{O}}/N_{\mathcal{O}}(\varepsilon)|\,w_\lambda(f+\varepsilon)$$
$$=q^{in-i(i-1)/2}\sum_{\varepsilon\in I_i}q^{-\sum_{j=1}^{n}\varepsilon_j j}w_\lambda(f+\varepsilon).$$

Set $\tilde{w}_\lambda(f)=q^{\sum_{i=1}^{n}(n-i)f_i}w_\lambda(f)$. We have shown that the function $\tilde{w}_\lambda$

on $\boldsymbol{Z}^n$ satisfies the following system of difference equations:

$$(6)\quad \begin{cases} \text{If } f_1 \geq f_2 \geq \cdots \geq f_n, \\ q^{i(i-1)/2}\lambda_i \tilde{w}_\lambda(f) = \sum_{\varepsilon \in I_i} \tilde{w}_\lambda(f+\varepsilon) \qquad (1 \leq i \leq n), \\ \text{If } f=(f_1, f_2, \cdots, f_n) \text{ does not satisfy the inequalities} \\ \quad f_1 \geq f_2 \geq \cdots \geq f_n,\ \tilde{w}_\lambda(f) = 0. \end{cases}$$

On the other hand, it is known (see e.g. [5]) that the function $\chi_f(\mu)$ given by (4), satisfies the following system of equations:

$$\chi_{f^i}(\mu)\chi_f(\mu) = \sum_{\varepsilon \in I_i} \chi_{f+\varepsilon}(\mu) \quad \text{if} \quad f_1 \geq f_2 \geq \cdots \geq f_n.$$

Our definition of μ implies $\chi_{f^i}(\mu) = q^{i(i-1)/2}\lambda_i$. Thus, as functions on $\boldsymbol{Z}^n$, $\tilde{w}_\lambda(f)$ and $\chi_f(\mu)$ satisfy the same system of difference equations (6). However, the solution of the equation system (6) is unique, up to a constant factor. Since $\tilde{w}_\lambda(0) = \chi_0(\mu) = 1$, we have $\tilde{w}_\lambda(f) = \chi_f(\mu)$.

References

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