0. "Whittaker functions" on \( \mathfrak{p} \)-adic linear groups have been studied by several authors (see e.g. [2] and [3]). In this note, we present an explicit formula for the class-1 "Whittaker functions" on \( GL_n(k) \), where \( k \) is a non archimedean local field.

1. Let \( k \) be a finite extension of the \( p \)-adic field \( \mathcal{O} \) and let \( \mathcal{O} \) be the ring of integers of \( k \). Choose a generator \( \pi \) of the maximal ideal of \( \mathcal{O} \) and denote by \( q \) the cardinality of the residue class field of \( k \). Set \( G=GL_n(k) \) and \( K=GL_n(\mathcal{O}) \). Then \( K \) is a maximal compact open subgroup of \( G \). The invariant measure of \( G \) is normalized so that the total volume of \( K \) is equal to 1. Denote by \( L_0(G, K) \) the space of complex valued compactly-supported bi-\( K \)-invariant functions on \( G \). Then \( L_0(G, K) \) is a commutative subalgebra of the group ring \( L^*(G) \) of \( G \).

We denote by \( N \) the group of \( n \times n \) upper triangular unipotent matrices with entries in \( k \). Choose a character \( \psi \) of the additive group of \( k \) which is trivial on \( \mathcal{O} \) but not trivial on \( \pi^{-1}(\mathcal{O}) \). Denote by the same letter \( \psi \) the character of \( N \) given by \( \psi(x)=\prod_{i=1}^{n-1} \psi(x_{i,i+1}) \), where \( x_{i,i+1} \) is the \((i,i+1)\)-entry of \( x \).

For each algebra homomorphism \( \lambda \) of \( L_0(G, K) \) into \( \mathbb{C} \), it is known that there uniquely exists a function \( W_\lambda(g) \) on \( G \) which satisfies the following conditions (1), (2) and (3).

\[
\begin{align*}
(1) \quad W_\lambda(xg) = \psi(x)W_\lambda(g) \quad (\forall x \in N), \\
(2) \quad \int_G W_\lambda(gx)\varphi(x)dx = \lambda(\varphi)W_\lambda(g) \quad (\forall \varphi \in L_0(G, K)), \\
(3) \quad W_\lambda(1) = 1.
\end{align*}
\]

The function \( W_\lambda \) is said to be the class-1 "Whittaker function" on \( G \) associated with the homomorphism \( \lambda \) of \( L_0(G, K) \) into \( \mathbb{C} \).

For each \( n \)-tuple \( f=(f_1, f_2, \ldots, f_n) \) of integers, we denote by \( \pi^f \) the diagonal matrix whose \( i \)-th diagonal entry is \( \pi^{f_i} \) \((i=1, \ldots, n)\). Set \( w_\lambda(f)=W_\lambda(\pi^f) \). It is known that \( G=\bigcup_{f \in \mathbb{Z}^n} N\pi^fK \) (disjoint union).

To evaluate \( W_\lambda \) on \( G \), it is sufficient to know \( w_\lambda(f) \) for all \( f \in \mathbb{Z}^n \), since \( W_\lambda \) is right \( K \)-invariant and satisfies (1). Since the conductor of \( \psi \) is \( \mathcal{O} \), it follows easily from (1) that \( w_\lambda(f) \) is zero unless \( f_1 \geq f_2 \geq \cdots \geq f_n \).

For \( i=1, 2, \ldots, n \), let \( \varphi_i \) be the characteristic function of the double
$K$-coset $K\pi^tK$, where $f^t=(1,1,\ldots,1,0,0,\ldots,0)$. It is known that $L_0(G,K)$ is isomorphic to the polynomial ring generated by $\varphi_1,\varphi_2,\ldots,\varphi_n$. Set $\lambda_i=\lambda(\varphi_i), (i=1,2,\ldots,n)$ and choose $n$ complex numbers $\mu_1,\mu_2,\ldots,\mu_n$ so that the $i$-th elementary symmetric function of $\mu_i's$ is equal to $q^{i(1-n)\lambda_i} (i=1,2,\ldots,n)$. Let $\mu$ be the diagonal matrix whose $i$-th diagonal entry is $\mu_i$ for $i=1,2,\ldots,n$. Since $\lambda_n\neq0$, $\mu\in GL_n(C)$.

For $f=(f_1,f_2,\ldots,f_n)\in Z^n$, denote by $\chi_f$ the character of the irreducible representation of $GL_n(C)$ with the highest weight $f$, if $f_1\geq f_2\geq\cdots\geq f_n$. Unless $f_1\geq f_2\geq\cdots\geq f_n$, set $\chi_f=0$.

**Theorem.** Notations and assumptions being as above, we have,

$$W_f(\pi^t)=q^\sum_{i=1}^n (i-n)f_i \chi_f(\mu) \quad (f\in Z^n),$$

where

$$\chi_f(\mu)=\begin{pmatrix}
\mu_1^{f_1+n-1} & \mu_2^{f_2+n-1} & \cdots & \mu_n^{f_n+n-1} \\
\vdots & \vdots & & \vdots \\
\mu_1^{f_1} & \mu_2^{f_2} & \cdots & \mu_n^{f_n} \\
\prod_{i<j} (\mu_i-\mu_j), & \text{if } f_1\geq\cdots\geq f_n \\
0, & \text{otherwise.}
\end{pmatrix}$$

**Proof.** We first prove the following sublemma:

Sublemma. (See Lemma 11 of [4].) Set $N_0=N\cap K$ and denote by $I_i$ the set of all the $n$-tuples $e=(e_1,\ldots,e_n)$ of non-negative integers which satisfy $e_1+e_2+\cdots+e_n=i$. Further, set $N_0(e)=N_0\cap \pi^tK\pi^{-t}$. Then we have

$$K\pi^tK= \bigcup_{e\in I_i, x\in N_0/N_0(e)} x\pi^tK \quad \text{(disjoint union).}$$

**Proof.** Set $e=(0,0,\ldots,0,1,1,\ldots,1)$. Denote by $B$ the subgroup of $K$ consisting of all matrices in $K$ whose subdiagonal entries are all in $\pi\mathcal{O}$. It is known (see [1]) that $K=\bigcup_{w\in W} BwB$ (disjoint union), where $W$ is the group of all permutation matrices in $K$. Since $(\pi\mathcal{O})^{-1}B(\pi\mathcal{O})\subset K\pi^tK$ and $Bw\pi^{-t}w^{-1}\subset \bigcup_{e\in I_i} N_0\pi^tK (\forall w\in W)$, $K\pi^tK=K\pi^tK$, $K\pi^tK$ is a subset of the right. Since the inverse inclusion relation is obvious, we obtain the sublemma.

It follows from the sublemma that if $f=(f_1,f_2,\ldots,f_n)\in Z^n$ and $f_1\geq f_2\geq\cdots\geq f_n$,

$$\lambda_1w_1(f)=\int_{\pi^tK} W_1(\pi^t x)\varphi_1(x)dx=\sum_{e\in I_i} |N_0/N_0(e)| w_1(f+e)$$

$$=q^{-n-i(1-n)/2} \sum_{e\in I_i} q^{-\frac{n}{2}}q^{\sum_{i=1}^n f_i} w_1(f+e).$$

Set $\tilde{w}_1(f)=q^{\sum_{i=1}^n (i-n)f_i} w_1(f)$. We have shown that the function $\tilde{w}_1$
on $\mathbb{Z}^n$ satisfies the following system of difference equations:

\[
\begin{align*}
  \begin{cases}
    \text{If } f_1 \geq f_2 \geq \cdots \geq f_n, \\
    q^{\ell_{i+1}} \overline{W}_i(f) = \sum_{r \in F_i} \overline{W}_i(f + r) \quad (1 \leq i \leq n), \\
    \text{If } f = (f_1, f_2, \cdots, f_n) \text{ does not satisfy the inequalities} \\
    f_1 \geq f_2 \geq \cdots \geq f_n, \overline{W}_i(f) = 0.
  \end{cases}
\end{align*}
\]

On the other hand, it is known (see e.g. [5]) that the function $\chi_f(\mu)$ given by (4), satisfies the following system of equations:

\[
\chi_f(\mu) = \sum_{\ell \in F_i} \chi_{f, \ell}(\mu) \quad \text{if} \quad f_1 \geq f_2 \geq \cdots \geq f_n.
\]

Our definition of $\mu$ implies $\chi_f(\mu) = q^{\ell(\ell-1)/2} \lambda_\ell$. Thus, as functions on $\mathbb{Z}^n$, $\overline{W}_i(f)$ and $\chi_f(\mu)$ satisfy the same system of difference equations (6).

However, the solution of the equation system (6) is unique, up to a constant factor. Since $\overline{W}_i(0) = \chi_0(\mu) = 1$, we have $\overline{W}_i(f) = \chi_f(\mu)$.

References