47. An Alternate Proof of a Transfer Theorem without using Transfer

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(Comm. by Kenjiro SHODA, M. J. A., April 12, 1976)

In the paper [1] by the same author, he proved

Theorem A. *If a Sylow p-subgroup P of a finite group G has no quotient group isomorphic to the wreath product $Z_p \wr Z_p$, where Z_p is the cyclic group of order p, then $P \cap G' = P \cap N_G(P)'$.*

The purpose of this paper is to give a primitive proof of a particular case of this theorem. Namely, we shall prove

Theorem B. *If a Sylow 2-subgroup P of a finite group G has no quotient group isomorphic to the dihedral group D_8 of order 8, then $P \cap G^2G' = P \cap N^2N'$, where $N = N_G(P)$. In particular, if G has no subgroup of index 2, then so does N.*

Most of the notation is standard. Let G be a finite group. Then G' denotes the commutator group of G. For $X \subseteq G$, $\langle X \rangle$ is the subgroup generated by X. We set $G^2G' = \langle g^2, G' \mid g \in G \rangle$. We write $H \lhd G$ if H is a normal subgroup of G. For subgroups H, K of G, the notation $K \backslash H$ denotes the set $\{Kh \mid h \in H\}$. Clearly, every element of H induces a permutation on $K \backslash H$. We write $H < G$ if H is a proper subgroup of G.

The following lemma is essential to the proof of Theorem B.

Lemma. *Let P be a 2-group, $K < S < P$ and $x \in P$. Assume the following:*

(a) $|S : K| = 2$;

(b) *For any $u \in P$, $\langle x^2 \rangle^u \cap S \subseteq K$;*

(c) *The element x acts on the set $K \backslash P$ as an odd permutation.*

Then P has a quotient group isomorphic to D_8.

Proof. We shall argue by induction on $|P : S|$. Let R be a subgroup of P such that $|R : S| = 2$. Suppose $K \lhd R$. Since x acts on $K \backslash P$ as an odd permutation, we have that there is $u \in P$ such that x acts as an odd permutation on the set $K \backslash Ru\langle x \rangle$. Replacing x with uxu^{-1}, we may assume that $u = 1$. If x fixes an element of $K \backslash R\langle x \rangle$, then x acts trivially on $K \backslash R\langle x \rangle$, as $K \lhd R$, a contradiction. Thus x acts semiregularly on $K \backslash R\langle x \rangle$, and so the number of the $\langle x \rangle$-orbits of $K \backslash R\langle x \rangle$ is 1 or 3. It follows easily from $K \lhd R$ that $K \backslash R\langle x \rangle = K \backslash K\langle x \rangle$. Thus $|\langle x \rangle \cap R : \langle x \rangle \cap K| = 4$. This means that $x^j \in S - K$ for some even j. This contradicts the assumption of this lemma. Hence we proved that K is not normal in R.

Let $r \in R-S$ and $N=K \cap K^r$. Then $N \lhd R$ and $R/N \cong D_8$ and $S/N \cong Z_2^2$. We may assume that $r^2 \in N$. Let L be a maximal subgroup of R such that $N < L \neq S$ and $L/N \cong Z_2^2$. We shall first show $\langle x^2 \rangle^u \cap R \leq L$. Let $y \in \langle x^2 \rangle^u \cap R, u \in P$. Then $y^2 \in \langle x^2 \rangle^u \cap S \leq K$ by the assumption of this lemma. Since $y^r \in \langle x^2 \rangle^{ur} \cap R$, we have similarly that $y^{2r} \in \langle x^2 \rangle^{ur} \cap S \leq K$. Thus $y^2 \in K \cap K^r = N$. If $y \in S$, then it follows from the assumption of this lemma that $y \in K \cap K^r = N \leq L$. If $y \notin S$, then $y \in L$, as $R/N \cong D_8$. Hence we have that $\langle x^2 \rangle^u \cap R \leq L$ for any $u \in P$.

Next, we will show that x acts on the set $L \backslash P$ as an odd permutation. Since x acts on $K \backslash P$ as an odd permutation, it will suffice to show that for each $u \in P$, the following are equivalent:

(i) x acts as an odd permutation on $L \backslash Ru\langle x \rangle$;

(ii) x acts as an odd permutation on $K \backslash Ru\langle x \rangle$.

If necessarily replacing x with uxu^{-1}, we may assume that $u=1$. Let $k \in K-L$ and $s=[r, k] \in L \cap S-N$. Then $R=L+Lk$.

Suppose $Rx \neq R$. If $Lx^j = Lk$ for some j, then $Rx^j = R$, and so $x^j \in R$. As $x \notin R, j$ is even. Thus $x^j \in L$ and so $Lx^j = L \neq Lk$, a contradiction. Hence $L\langle x \rangle \neq Lk\langle x \rangle$. In particular, x is represented on $L \backslash R\langle x \rangle$ as the product of two nontrivial cyclic permutation, and hence x acts on $L \backslash R\langle x \rangle$ as an even permutation. If $Kx^j = Ks$ for some j, then $Rx^j = R$, so j is even. Thus $x^j \in K$ by the assumption of this lemma, a contradiction. Thus $K\langle x \rangle \neq Ks\langle x \rangle$. Since $rK = Krs$, we conclude that x acts an even permutation on $K \backslash R\langle x \rangle$. Thus in this case, neither (i) nor (ii) holds.

Suppose next $Rx = R$. Then $x \in R$. Since $x^2 \in S \lhd R$, we have that $x^2 \in N$ by the assumption of this lemma, and so $x \in S \cup L$. Thus (i) is equivalent to

(i)′ $$x \in L-S = N\langle s \rangle k.$$

If $x \in Nk$, then $x: K \rightarrow K, Ks \rightarrow Ks, Kr \leftrightarrow Krs$. If $x \in Nsk$, then $x: K \leftrightarrow Ks, Kr \rightarrow Kr, Krs \rightarrow Krs$. Thus if (i) holds, then (ii) also holds. Assume conversely that (ii) holds. Then $x \notin N$ and x fixes an element of $K \backslash R$, and so $x \in K \cup K^r - N = N\langle s \rangle k$. Hence (i)′ and also (i) hold. We proved that (i) and (ii) are equivalent in this case.

We can now prove this lemma. We show that $\langle x^2 \rangle^u \cap R \leq L$ for any $u \in P$ and that x acts on the set $L \backslash P$ as an odd permutation. So if $R \neq P$, then we can apply induction and hence P has a quotient group isomorphic to D_8. If $R = P$, then we already proved that $R/N = P/N \cong D_8$. The lemma is proved.

Proof of Theorem B. Let G, P, N be as in the theorem. Suppose the theorem is false. Then $P \cap G^2G' \neq P \cap N^2N'$. Take an element x of $P \cap G^2G' - N^2N'$ of minimal order. There is a subgroup M of

$N=N_G(P)$ of index 2 such that $x\notin M$. As $x\in G^2G'$, the element x acts on the set $M\backslash G$ as an even permutation. Since $x: M\leftrightarrow Mx=N-M$, there is $g\in G-N$ such that x acts on $M\backslash NgP$ as an odd permutation. As $M\lhd N$, we see that the permutation representations $(P, M\backslash MgP)$ and $(P, M\backslash MxgP)$ are equivalent. Since x acts on $M\backslash NgP$ as an odd permutation, we have that $MgP=MxgP=NgP$. Set $S=P\cap N^g$ and $K=P\cap M^g$. As $M^gP=N^gP$, we have that $|S:K|=2$. As $g\notin N=N_G(P)$, $S<P$. If $u\in P$, then by the minimality of the order of x, we have that $\langle x^2\rangle^{ug^{-1}}\cap P\leq M$, and so $\langle x^2\rangle^u\cap S\leq K$. Furthermore, the permutation repreaentations $(P, M\backslash MgP)$ and $(P, K\backslash P)$ are equivalent, so x acts on $K\backslash P$ as an odd permutation. By Lemma 1, we have that P has a quotient group isomorphic to D_8, contrary to the assumption of the theorem about P. The theorem is proved.

Reference

[1] T. Yoshida: Character-theoretic transfers (to appear).