45. On Quadratic Differentials with Closed Trajectories and their Applications

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In this paper we shall consider on compact Riemann surfaces the class of quadratic differentials with finite norm and closed trajectories and state some theorems, the detailed proofs of which will be given in another paper ([5]) together with related results.

1. Let R be a compact Riemann surface of genus g (>0), we write

$\tilde{A}_1=\tilde{A}_1(R)=\{\theta: \theta$ is a holomorphic abelian differential on $R.\}$

$\tilde{A}_2=\tilde{A}_2(R)=\{\theta: \theta$ is a holomorphic quadratic differential on $R.\}$

$A_2D=A_2D(R)=\Big\{\phi: \phi$ is a meromorphic quadratic differential on R such that $|||\phi|||=\int_R |\phi|<+\infty.\Big\}$

$CA_2D=CA_2D(R)=\{\phi\in A_2D$ with closed trajectories.$\}$

$C\tilde{A}_1=C\tilde{A}_1(R)=\{\theta\in\tilde{A}_1: \theta^2\in CA_2D.\}$

$C\tilde{A}_2=C\tilde{A}_2(R)=CA_2D\cap\tilde{A}_2.$

Then we can prove the following

Proposition 1 (cf. [1]). *Let γ be an arbitrary 1-cycle on R. Then the holomorphic reproducing differential θ_γ for γ belongs to $C\tilde{A}_1$.*

And using this proposition, we have

Proposition 2. *The set $C\tilde{A}_1$ is dense in $\tilde{A}_1$ with respect to the Dirichlet norm.*

Thus making two (or four if necessary) sheeted covering surface R' of R, and considering the class of odd holomorphic reproducing differentials on R', we can prove the following Strebel's conjecture ([3]).

Theorem 1. *The set CA_2D is dense in A_2D with respect to the $|||\ |||$-norm. Moreover if $\phi\in A_2D$ has poles at $\{P_i\}_{i=1}^r$, then ϕ can be approximated by the elements of CA_2D with poles at $\{P_i\}_{i=1}^r$.*

The last assertion follows from the fact that the norm convergence is equivalent to the locally uniform convergence.

2. The set of holomorphic reproducing differentials is dense in $\tilde{A}_1$. But an element of $C\tilde{A}_1$ is not always proportional to some holomorphic reproducing differential in the case that $g\geqq 2$. This follows from the following

Theorem 2. *There exists an admissible curve system* $\{\gamma_i\}$ *such that* $D\{\gamma_i\}=\{\theta \in C\tilde{A}_1$: *the moduli vector of* θ^2 *belongs to the moduli surface of* $\{\gamma_i\}$.$\}$ (cf. [2], [5]) *contains a real g-dimensional manifold, which is contained in*

$$P\{A_j\}=\left\{\theta \in \tilde{A}_1 : \theta=\sum_{j=1}^{g} c_j\theta_{A_j} \text{ for real } c_j.\right\}$$

for suitable normal homology base $\{A_j, B_j\}_{j=1}^{g}$ *on* R.

Here we remark that for every admissible curve system $\{\gamma_i\}$ on R the set $D\{\gamma_i\}$ is contained in a real g-dimensional manifold.

3. Let T_g be the Teichmüller space of compact Riemann surfaces of genus g (>0), R_0 and R be points of T_g, and $B_{R_0}(R)$ and $D_{R_0}(R)$ be the Beltrami coefficient and the maximal dilatation of the Teichmüller mapping from R_0 to R, respectively. It is well known that there exist a $\phi \in \tilde{A}_2(R_0)$ and k $(1>k\geqq 0)$ such that

$$B_{R_0}(R)=k\frac{\bar{\phi}}{|\phi|}, \qquad D_{R_0}(R)=K=\frac{1+k}{1-k}(\geqq 1).$$

Now let $C_R(\phi)$ be the Strebel's contraction of $\phi \in C\tilde{A}_2(R_0)$ from R_0 to R (cf. [4]). Then we can characterize $B_{R_0}(R)$ by this contractions.

Theorem 3. *It holds that*

$$D_{R_0}(R)=\sup_{\phi\in C\tilde{A}_2(R_0)} C_R(\phi), \qquad \frac{1}{D_{R_0}(R)}=\inf_{\phi\in C\tilde{A}_2(R_0)} C_R(\phi).$$

Theorem 4. *Let* $B_{R_0}(R)=k\frac{\bar{\phi}}{|\phi|}$ *and* $|||\phi|||=1$. *If a sequence* ϕ_n *in* $C\tilde{A}_2(R_0)$ *with* $|||\phi_n|||=1$ *satisfies the condition that* $\lim_{n\to\infty} C_R(\phi_n)=\frac{1}{K}$ *with* $K=\frac{1+k}{1-k}$, *then* ϕ_n *converges to* ϕ *with respect to the* $|||\ \ |||$*-norm.*

As a corollary, if $C_R(\phi)\geqq 1$ for every $\phi \in C\tilde{A}_2(R_0)$, then R is conformally equivalent to R_0.

References

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