60. *Scalar Extension of Quadratic Lattices*

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Let E/F be a finite extension of algebraic number fields, $\mathcal{O}_E, \mathcal{O}_F$ the maximal orders of E, F respectively. Let L, M be quadratic lattices over $\mathcal{O}_F$ in regular quadratic spaces U, V over F respectively; then we are concerned about the following question:

We assume:

(*) there is an isometry σ from $\mathcal{O}_E L$ onto $\mathcal{O}_E M$,

where $\mathcal{O}_E L, \mathcal{O}_E M$ denote tensor products of $\mathcal{O}_E$ and L, M over $\mathcal{O}_F$ respectively.

Does the assumption imply $\sigma(L)=M$?

The answer is negative if a quadratic space $EU(\cong EV)$ is indefinite. Even if we suppose that EU is definite, the answer is negative in general. However it seems to be affirmative if we confine ourselves to the following cases:

F: the field $\boldsymbol{Q}$ of rational numbers,

E: a totally real algebraic number field,

L, M: definite quadratic lattices over the ring $\boldsymbol{Z}$ of rational integers.

We give some evidences here. Detailed proofs will appear elsewhere.

Theorem 1. *Let m be an integer ≥ 2, and E be a totally real algebraic number field with degree m, and assume that L, M be definite quadratic lattices over $\boldsymbol{Z}$. Then the assumption (*) implies $\sigma(L)=M$, if E does not contain a finite number of (explicitly determined) algebraic integers whicn are not dependent on L, M, but on m.*

Theorem 2. *Let E be totally real, and L, M be binary or ternary definite quadratic lattices over $\boldsymbol{Z}$. The assumption (*) implies $\sigma(L)=M$.*

Corollary. *Let E, K be a totally real algebraic number field and an imaginary quadratic field respectively whose discriminants are relatively prime. Then an ideal of K is principal if it is principal in a composite field KE.*

Theorem 3. *Let E be a real quadratic, totally real cubic or totally real Dirichlet's biquadratic field, and L, M be definite quadratic lattices over $\boldsymbol{Z}$. Then the assumption (*) implies $\sigma(L)=M$.*

In case that $L=M$ and σ gives an orthogonal decomposition of

$\mathcal{O}_E L$, we have

Theorem 4. *Let E/F be a Galois extension of totally real algebraic number fields. Assume that an intermediate field K between E and F is F if K is unramified over F. If a definite quadratic lattice L over $\mathcal{O}_F$ is decomposable over $\mathcal{O}_E$, $\mathcal{O}_E L = L'_1 \perp \cdots \perp L'_m$, then there is a decomposition of $L = L_1 \perp \cdots \perp L_m$ with $L'_i = \mathcal{O}_E L_i$, in other words, a definite indecomposable quadratic lattice over $\mathcal{O}_F$ remains indecomposable over $\mathcal{O}_E$.*

Corollary. *Let E be a totally real algebraic number field, and L be a definite indecomposable quadratic lattice over $\mathbf{Z}$. Then $\mathcal{O}_E L$ is also indecomposable.*

Remark. Let E/F be an unramified extension of totally real algebraic number fields. Then there exists a definite indecomposable quadratic lattice over $\mathcal{O}_F$ which is decomposable over $\mathcal{O}_E$.

Our question is closely related to the problem:

If $\mathcal{O}_E L, \mathcal{O}_E M$ are isometric, then are L, M isometric?

In case of similar problems for spinor genus, see [1].

To prove our results the behaviour of the minimum under the scalar extension is investigated.

Added in the proof. Recently Theorem 2.3 were fairly improved.

References

[1] A. G. Earnest and J. S. Hsia: Springer-type theorems for spinor genera of quadratic forms. Bull. Amer. Math. Soc., **81**, 942–943 (1975).

[2] O. T. O'Meara: Introduction to Quadratic Forms. Springer-Verlag, (1963).