79. *On the Singularities of the Riemann Functions of Mixed Problems for the Wave Equation in Plane-Stratified Media. I*

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1. Introduction. The study on the singularities of the fundamental solutions (or Riemann functions) of mixed initial boundary value problems for linear hyperbolic equation with constant coefficients in a quarter space has been developed primarily by Duff [2] and afterward by Deakin [1], Matsumura [6], Wakabayashi [9], Tsuji [8], especially by Wakabayashi [10] and [11]. The purpose of this series of notes is to show that the methods in [6], [8]–[11] are applicable to the study of the singularities of the Riemann functions of mixed initial boundary value problems with a plane interface in a quarter space for the wave equation. This problem was suggested by Wilcox [12].

2. Formulation of the problem and Lopatinski's determinant. $\boldsymbol{R}^n$ denotes the n-dimensional Euclidean space and $\boldsymbol{\Xi}^n$ denotes its real dual space with duality $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$. Let us write $x' = (x_1, \cdots, x_{n-1})$, $x'' = (x_2, \cdots, x_n)$ for the coordinate $x = (x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$ and $\xi' = (\xi_1, \cdots, \xi_{n-1})$, $\xi'' = (\xi_2, \cdots, \xi_n)$ for the dual coordinate $\xi = (\xi_1, \cdots, \xi_n)$ in $\boldsymbol{\Xi}^n$. x_1 will play the role of time variable and x'' will play the role of physical space variable. Let h be a given positive number, and set $\Omega_I = \{x'' \in \boldsymbol{R}^{n-1}; 0 < x_n < h\}$ and $\Omega_{II} = \{x'' \in \boldsymbol{R}^{n-1}; x_n > h\}$. We consider two wave operators $P_1(D) = a_1^2\Delta - D_1^2$ and $P_2(D) = a_2^2\Delta - D_1^2$ with wave speeds $a_1 > 0$ and $a_2 > 0$ which govern the wave propagation in Ω_I and Ω_{II}, respectively. Here $D_j = \partial/i\partial x_j$ and $\Delta = D_2^2 + \cdots + D_n^2$. The mixed problem we will study is

(1) $P_1(D)u(x) = f(x),\ x_1 > 0,\ 0 < x_n < h$ (i.e. in $\boldsymbol{R}_+ \times \Omega_I$),

(2) $P_2(D)u(x) = f(x),\ x_1 > 0,\ x_n > h$ (i.e. in $\boldsymbol{R}_+ \times \Omega_{II}$),

(3) $u(0, x'') = g_0(x''),\ (D_1u)(0, x'') = g_1(x'')$ (initial conditions),

(4) $Q(D)u(x)|_{x_n=0} = k_0(x'),\ x_1 > 0$ (boundary condition)

(5) $B_j(D)u(x)|_{x_n=h-} = C_j(D)u(x)|_{x_n=h+} + k_j(x'),\ x_1 > 0,\ j = 1, 2$ (interface or transmission conditions),

where $Q(D)$, $B_j(D)$ and $C_j(D)$ are partial differential operators with constant coefficients.

Let Γ_ι denote the cone $\{\eta \in \boldsymbol{\Xi}^n; \eta_1 > 0, \eta_1^2 > a_\iota^2|\eta''|^2\}$ and let us denote by $\lambda_\iota^+ = \lambda_\iota^+(\xi' + i\eta')$ and $\lambda_\iota^- = \lambda_\iota^-(\xi' + i\eta')$ the roots with positive and negative imaginary part, respectively, of the equation in λ: $P_\iota(\xi'+i\eta';\lambda)=0$ for $\xi'+i\eta'\in\Xi^{n-1}-i\dot{\Gamma}_\iota$ $(\iota=1,2)$ where $\dot{\Gamma}_\iota=\{\eta'\in\Xi^{n-1};(\eta',0)\in\Gamma_\iota\}$. Now we set

$$(6)\qquad \begin{aligned} R_0(\xi'+i\eta')&=Q(\lambda_1^+)\begin{vmatrix} B_1(\lambda_1^-) & C_1(\lambda_2^+)\\ B_2(\lambda_1^-) & C_2(\lambda_2^+)\end{vmatrix},\\ R_1(\xi'+i\eta')&=Q(\lambda_1^-)\begin{vmatrix} B_1(\lambda_1^+) & C_1(\lambda_2^+)\\ B_2(\lambda_1^+) & C_2(\lambda_2^+)\end{vmatrix}\end{aligned}$$

and define the Lopatinski's determinant of the system $\{P_1(D),P_2(D),Q(D),B_j(D),C_j(D),j=1,2\}$ for the mixed problem (1)–(5) by

$$(7)\qquad \begin{aligned} R(\xi'+i\eta')=&R_0(\xi'+i\eta')\exp\{ih\lambda_1^-(\xi'+i\eta')\}\\ &-R_1(\xi'+i\eta')\exp\{ih\lambda_1^+(\xi'+i\eta')\}.\end{aligned}$$

Here $Q(\lambda_1^\pm)\equiv Q(\xi'+i\eta',\lambda_1^\pm(\xi'+i\eta'))$, $B_j(\lambda_1^\pm)\equiv B_j(\xi'+i\eta',\lambda_1^\pm(\xi'+i\eta'))$ and $C_j(\lambda_2^+)\equiv C_j(\xi'+i\eta',\lambda_2^+(\xi'+i\eta'))$. Note that $Q(\lambda_1^+)$ is the Lopatinski's determinant of the system $\{P_1(D),Q(D)\}$ for the mixed problem in the quarter space $\{x\in\boldsymbol{R}^n;x_1>0,x_n>0\}$ and $\begin{vmatrix} B_1(\lambda_1^-) & C_1(\lambda_2^+)\\ B_2(\lambda_1^-) & C_2(\lambda_2^+)\end{vmatrix}(\exp\{ih(\lambda_1^-+\lambda_2^+)\})$ is that of the system $\{P_1(D),P_2(D),B_j(D),C_j(D),j=1,2\}$ for the transmission problem in the space $\{x\in\boldsymbol{R}^n,x_1>0\}$ with the plane interface $x_n=h$ (see Hersh [3]).

Theorem. *The mixed problem* (1)–(5) *is* C^∞ (*or* $\mathcal{E}$) *well posed if and only if the following conditions*[1] *are satisfied.*

(8) $R_0^0(\vartheta')\neq 0,\qquad \vartheta=(1,0\cdots,0),$

(9) $R_0(\xi'-i\gamma\vartheta')=R_0(\xi_1-i\gamma,\xi_2,\cdots,\xi_n)\neq 0$ *for any* $\xi'\in\Xi^{n-1}$ *and* $\gamma\geqslant\gamma_1$, *where* R_0^0 *is the principal part of* R_0 (See Sakamoto [7]).

Making use of the Tarski-Seidenberg theorem, we can deduce from (9) that the following inequality holds for some constants M_1 and M_2.

$$(10)\qquad |R_0(\xi'-i\gamma\vartheta')|\geqslant M_1(1+|\xi'|+|\gamma|)^{M_2},\quad \xi'\in\Xi^{n-1},\quad \gamma\geqslant\gamma_1.$$

Since we have $|R_1(\xi'-i\gamma\vartheta')|\leqslant M_3(1+|\xi'|+|\gamma|)^{M_4}$ and $|\exp\{2ih\lambda_1^+(\xi'-i\gamma\vartheta')\}|\leqslant\exp\{-2h\gamma/a_1\}$, there exists a constant δ such that

$$(11)\qquad \begin{aligned} &|R_1(\xi'-i\gamma\vartheta')\exp\{2ih\lambda_1^+(\xi'-i\gamma\vartheta')\}/R_0(\xi'-i\gamma\vartheta')|\leqslant\delta<1,\\ &\qquad\qquad\qquad\qquad\qquad \xi'\in\Xi^{n-1},\qquad \gamma\geqslant\gamma_1,\end{aligned}$$

if we take $\gamma=m\log(2+|\xi'|)$. Here m is a positive real large enough. Hence we have

$$(12)\qquad \begin{aligned} &R(\xi'-i\gamma\vartheta')\\ &=R_0(\xi'-i\gamma\vartheta')(1-[R_1(\xi'-i\gamma\vartheta')\exp\{2ih\lambda_1^+(\xi'-i\gamma\vartheta')\}/R_0(\xi'-i\gamma\vartheta')])\\ &\neq 0,\qquad \xi'\in\Xi^{n-1},\qquad \gamma=m\log(2+|\xi'|).\end{aligned}$$

3. The Riemann function for the mixed problem. The solution of the mixed problem (1)–(5) can be represented in terms of the Riemann function and the Poisson kernels. In this note we will construct and study only the Riemann function, since the Poisson kernels can be constructed and studied in a similar way. Let y be an arbitrarily fixed

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1) The author was suggested these conditions by Dr. N. Iwasaki.

2) Note that $\lambda_1^-(\xi'+i\eta')=-\lambda_1^+(\xi'+i\eta')$.

point $(0, y'')$, $y'' \in \Omega_\iota$, $\iota = I$ or II. The Riemann function for the mixed problem (1)–(5) is defined as the unique distribution solution $G(x, y)$ of the mixed problem (1)–(5) with $f=0$, $g_0=0$, $g_1(x'')=\delta(x''-y'')$ and $k_0 = k_1 = k_2 = 0$, where $\delta(x)$ is the Dirac distribution. The Riemann function of the forward Cauchy problem for $P_\iota(D)$ in the whole physical space $\boldsymbol{R}^{n-1}$ is defined as the inverse Fourier-Laplace transform of $P_\iota(\xi + i\eta)^{-1}$ in the sense of distributions:

$$(13) \qquad E_\iota(x) = (2\pi)^{-n} \int_{\Xi^n} P_\iota(\xi + i\eta)^{-1} \exp\{i\langle x, \xi + i\eta\rangle\} d\xi$$

where $\eta \in -s\vartheta - \Gamma_\iota$ with a positive real s large enough. Then the distribution $E_\iota(x-y)$ describles the incident or primary propagation of waves due to a point source $\delta(x''-y'')$. We define the secondary Riemann function $F(x, y) = F_\iota(x, y)$ in Ω_ι in the following way. *Case* $0 < y_n < h$: $G(x, y) = E_1(x-y) - F_1(x, y)$ for $0 < x_n < h$ and $G(x, y) = F_2(x, y)$ for $x_n > h$. *Case* $y_n > h$: $G(x, y) = F_1(x, y)$ for $0 < x_n < h$ and $G(x, y) = E_2(x-y) - F_2(x, y)$ for $x_n > h$. $F(x, y)$ describes the propagation of secondary waves caused by the primary waves, the boundary wall $x_n = 0$ and the interface wall $x_n = h$. Thus the F_1 and F_2 are given as the solutions of the following equations. *Case* $0 < y_n < h$: $P_\iota(D_x)F_\iota(x, y) = 0$ for $x_1 > 0$, $x'' \in \Omega_\iota (\iota = 1, 2)$, $Q(D_x)F_1(x, y)|_{x_n=0} = Q(D_x)E_1(x-y)|_{x_n=0}$,

$$B_j(D_x)(E_1(x-y) - F_1(x, y))|_{x_n=h-} = C_j(D_x)F_2(x, y)|_{x_n=h+} \qquad (j=1, 2).$$

Case $y_n > h$: $P_\iota(D_x)F_\iota(x, y) = 0$ for $x_1 > 0$ and $x'' \in \Omega_\iota$ $(\iota = 1, 2)$,

$$Q(D_x)F_1(x, y)|_{x_n=0} = 0,$$

$$B_j(D_x)F_1(x, y)|_{x_n=h-} = C_j(D_x)(E_2(x-y) - F_2(x, y))|_{x_n=h+} \qquad (j=1, 2).$$

Taking thus formally partial Fourier-Laplace transforms with respect to $x' = (x_1, \cdots, x_{n-1})$ in these equations, we obtain a system of ordinary differential equations in x_n with coefficients depending on the parameter $\xi' + i\eta'$. From (12) we can find uniquely the solutions $\hat{F}_\iota(\xi' - i\gamma\vartheta', x_n, y)$, $\iota = 1, 2$ of the form: $\hat{F}_1 = a \exp\{ix_n\lambda_1^+\} + b \exp\{ix_n\lambda_1^-\}$, $\hat{F}_2 = c \exp\{ix_n\lambda_2^+\}$ if we take $\gamma = m \log(2 + |\xi'|)$. Let S_m be the surface $\{\xi = (\xi_1 - i\eta_1, \xi_2, \cdots, \xi_n),\ \xi \in \Xi^n,\ \eta_1 = m \log(2 + |\xi'|)$ in $\boldsymbol{C}^n$. Then $F_1(x, y)$ and $F_2(x, y)$ can be obtained by applying the inverse Fourier-Laplace transformation along S_m to the solutions $\hat{F}_1$ and $\hat{F}_2$. We shall give the more explicit expressions in the following note.

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