99. *Some Results on Additive Number Theory. II*

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In this note we outline the proof of the

Theorem. *Let k be an integer >1, and let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$. For sufficiently large positive integer N, let $A(N)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$\log\log N+\alpha_i\sqrt{\log\log N}<\omega(n_i)<\log\log N+\beta_i\sqrt{\log\log N}\quad(i=1,\cdots,k)$$

simultaneously, where $\omega(n_i)$ denotes the number of distinct prime factors of n_i. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-k/2}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-x^2/2}dx.$$

This theorem was announced as Theorem 3 in [2] without proof. Our proof is elementary and makes no use of any limit theorems in probability theory.

Lemma 1. *Let a_i $(i=1,\cdots,k)$ and b be positive integers such that $d=(a_1,\cdots,a_k)$ divides b. Let S denote the number of solutions of the Diophantine equation $a_1x_1+\cdots+a_kx_k=b$ in positive integers, then we have $|S-db^{k-1}/[(k-1)!\,a_1\cdots a_k]|<Cb^{k-2}$, where C is a positive number dependent only on k and independent of a_i and b.*

We define the set P_N consisting of primes as $P_N=\{p: e^{(\log\log N)^2}<p<N^{(\log\log N)^{-2}}\}$ and put $y(N)=\sum_{p\in P_N}1/p$. Then we have

$$y(N)=\log\log N+O(\log\log\log N). \tag{1}$$

We denote by $\omega_N(n)$ the number of primes p such that $p|n$, $p\in P_N$.

For any positive integer t, we define the set $M(t)$ consisting of positive integers as $M(t)=M(N;t)=\{m: m \text{ is squarefree}; m \text{ has } t \text{ prime factors}; p|m\Rightarrow p\in P_N\}$. We put for convenience $M(0)=\{1\}$.

For any k positive integers t_i, we denote by $F(N;t_1,\cdots,t_k)$ the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that $\omega_N(n_i)=t_i$ simultaneously.

For any k positive integers $m_i\in M(t_i)$ with some positive integers t_i, we denote by $G(N;m_1,\cdots,m_k)$ the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that $\prod_{p|n_i,p\in P_N}p=m_i$ simultaneously. We have

$$F(N;t_1,\cdots,t_k)=\sum_{m_1\in M(t_1)}\cdots\sum_{m_k\in M(t_k)}G(N;m_1,\cdots,m_k).$$

For any $k+1$ positive integers t_i and T, we put

$$\mathcal{H}^{(0)}(N\,;t_1,\cdots,t_k\,;T)=\sum_{m_1\in M(t_1)}\cdots\sum_{m_k\in M(t_k)}\mathcal{K}^{(0)}(N\,;m_1,\cdots,m_k\,;T),$$

$$\mathcal{K}^{(0)}(N\,;m_1,\cdots,m_k\,;T)=\sum_{\tau_1=0}^{2T}\cdots\sum_{\tau_k=0}^{2T}(-1)^{\tau_1+\cdots+\tau_k}\mathcal{L}(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k),$$

$$\mathcal{L}(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)=\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in M(\tau_k)\\(\mu_k,m_k)=1}}\sum_{\substack{n_1+\cdots+n_k=N\\m_i\mu_i|n_i\ (i=1,\cdots,k)}}1.$$

Similarly we put

$$\mathcal{H}^{(i)}(N\,;t_1,\cdots,t_k\,;T)=\sum_{m_1\in M(t_1)}\cdots\sum_{m_k\in M(t_k)}\mathcal{K}^{(i)}(N\,;m_1,\cdots,m_k\,;T),$$

$$\mathcal{K}^{(i)}(N\,;m_1,\cdots,m_k\,;T)$$
$$=\sum_{\tau_1=0}^{2T}\cdots\sum_{\tau_i=0}^{2T+1}\cdots\sum_{\tau_k=0}^{2T}(-1)^{\tau_1+\cdots+\tau_k}\mathcal{L}(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k),$$

where τ_i runs through $0,\cdots,2T+1$, and other τ's run through $0,\cdots,2T$.

Lemma 2. $\sum_{i=1}^{k}\mathcal{H}^{(i)}-(k-1)\mathcal{H}^{(0)}\leqq F\leqq\mathcal{H}^{(0)}$.

For brevity we write $\mathcal{H}^{(0)}$ etc. for $\mathcal{H}^{(0)}(N\,;t_1,\cdots,t_k\,;T)$ etc. Now we have

$$\mathcal{L}=\sum_{n_1+\cdots+n_k=N,\ m_i\mu_i|n_i}\prod_{i=1}^{k}\binom{\omega_N(n_i)-t_i}{\tau_i},$$

and, as in the proof of Lemma 3 in [1], we have

$$\sum_{i=1}^{k}\mathcal{K}^{(i)}-(k-1)\mathcal{K}^{(0)}\leqq G\leqq\mathcal{K}^{(0)},$$

from which the lemma follows.

We shall use this lemma to obtain a certain asymptotic formula for F by proving easier ones for $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(i)}$ giving T an appropriate value. This procedure might be said to be a type of sieve method. As $\mathcal{H}^{(i)}$ can be dealt with in almost the same way as $\mathcal{H}^{(0)}$, we shall be concerned with $\mathcal{H}^{(0)}$. For this purpose we introduce some more functions. We put

$$H_j(N\,;t_1,\cdots,t_k\,;T)=\sum_{m_1\in M(t_1)}\cdots\sum_{m_k\in M(t_k)}K_j(N\,;m_1,\cdots,m_k\,;T),$$

$$K_j(N\,;m_1,\cdots,m_k\,;T)$$
$$=\sum_{\tau_1=0}^{2T}\cdots\sum_{\tau_k=0}^{2T}(-1)^{\tau_1+\cdots+\tau_k}L_j(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k),$$

$$L_0(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)=\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in M(\tau_k)\\(\mu_k,m_k)=1\\(m_1\mu_1,\cdots,m_k\mu_k)|N}}\frac{(m_1\mu_1,\cdots,m_k\mu_k)}{m_1\mu_1\cdots m_k u_k},$$

$$L_1(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)=\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in M(\tau_k)\\(\mu_k,m_k)=1\\(m_1\mu_1,\cdots,m_k\mu_k)=1}}\frac{1}{m_1\mu_1\cdots m_k\mu_k},$$

$$L_2(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)$$
$$=\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1\\(m_1\mu_1,\cdots,m_k\mu_k)>1,}}\cdots\sum_{\substack{\mu_k\in M(\tau_k)\\(\mu_k,m_k)=1\\(m_1\mu_1,\cdots,m_k\mu_k)|N}}\frac{(m_1\mu_1,\cdots,m_k\mu_k)}{m_1\mu_1\cdots m_k\mu_k},$$

$$L_3(N\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)=\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in M(\tau_k)\\(\mu_k,m_k)=1\\(m_1\mu_1,\cdots,m_k\mu_k)>1}}\frac{1}{m_1\mu_1\cdots m_k\mu_k}.$$

Now, from the above definitions, we at once have

$$(2)\qquad H_0=H_1+H_2,$$

$$(3)\qquad H_1+H_3=\prod_{i=1}^{k}\sum_{m_i\in M(t_i)}\frac{1}{m_i}\sum_{\tau_i=0}^{2T}(-1)^{\tau_i}\sum_{\mu_i\in M(\tau_i),\,(\mu_i,m_i)=1}\frac{1}{\mu_i}.$$

Lemma 3. *Let $T=[4y(N)]+1$. Then, as $N\to\infty$, we have*

$$H_1+H_3=(t_1!\cdots t_k!)^{-1}\{y(N)\}^{t_1+\cdots+t_k}e^{-ky(N)}\{1+o(1)\}$$

uniformly in t_i with $t_i<2y(N)$ simultaneously.

We consider the k factors on the right-hand side of (3) separately. By similar reasoning as in the proofs of Lemmas 4 and 5 in [1], we can see that

$$\sum_{m_i\in M(t_i)}\frac{1}{m_i}\sum_{\tau_i=0}^{2T}(-1)^{\tau_i}\sum_{\mu_i\in M(\tau_i),\,(\mu_i,m_i)=1}\frac{1}{\mu_i}=\frac{\{y(N)\}^{t_i}e^{-y(N)}}{t_i!}\{1+o(1)\}$$

uniformly in t_i with $t_i<2y(N)$, which gives the lemma.

Lemma 4. *As $N\to\infty$, we have*

$$H_2=o[(t_1!\cdots t_k!)^{-1}\{y(N)\}^{t_1+\cdots+t_k}e^{-ky(N)}]$$

uniformly in t_i with $t_i<2y(N)$ and in arbitrary T, and similarly for H_3.

For each summand of the sum defining L_2, we put $d=(m_1\mu_1,\cdots,m_k\mu_k)$, $m_i\mu_i=dm_i'\mu_i'$ with $m_i'|m_i$, $\mu_i'|\mu_i$. Then, since $(m_i,\mu_i)=1$, it follows that

$$(4)\qquad |H_2|\leqq\sum_d\frac{1}{d}\cdot\Big(\prod_{i=1}^{k}\sum_{t_i=0}^{\infty}\sum_{m_i\in M(t_i)}\frac{1}{m_i}\Big)^2,$$

and this inequality remains true, when we let d run through the square-free integers >1 such that the number of prime factors of d is $<\log N$ and each prime factor of d is $>e^{(\log\log N)^2}$. Now easy calculations give

$$(5)\qquad \textstyle\sum_d 1/d=O(e^{-(\log\log N)^2}\log N).$$

On the other hand

$$(6)\qquad \sum_{t_i=0}^{\infty}\sum_{m_i\in M(t_i)}\frac{1}{m_i}\leqq\sum_{t_i=0}^{\infty}\frac{\{y(N)\}^{t_i}}{t_i!}=e^{y(N)}.$$

It follows from (4), (5) and (6) that

$$H_2=O(e^{2ky(N)-(\log\log N)^2}\log N).$$

From this and (1), we obtain the desired estimation for H_2. H_3 can be treated similarly.

Lemma 5. *Let $T=[4y(N)]+1$. Then, as $N\to\infty$, we have*

$$H_0=(t_1!\cdots t_k!)^{-1}\{y(N)\}^{t_1+\cdots+t_k}e^{-ky(N)}\{1+o(1)\}$$

uniformly in t_i with $t_i<2y(N)$.

The lemma follows from (2) and Lemmas 3 and 4.

Lemma 6. *Let $T=[4y(N)]+1$. Then, as $N\to\infty$, we have*

$$\mathcal{H}^{(0)}=\{(k-1)!\,t_1!\cdots t_k!\}^{-1}N^{k-1}\{y(N)\}^{t_1+\cdots+t_k}e^{-ky(N)}\{1+o(1)\}$$

uniformly in t_i with $t_i<2y(N)$, and similarly for $\mathcal{H}^{(i)}$.

By Lemma 1,

$$(7)\qquad \mathscr{H}^{(0)}-N^{k-1}H_0/(k-1)!=O\left\{N^{k-2}\left(\prod_{i=1}^{k}\sum_{t_i=0}^{2T}\sum_{m_i\in M(t_i)}1\right)^2\right\},$$

since $t_i<T$. Also, since $T<5\log\log N$ for large N by (1), we can see that

$$\sum_{t_i=0}^{2T}\sum_{m_i\in M(t_i)}1=O(N^{10(\log\log N)^{-1}}).$$

It follows from this and (7) that

$$\mathscr{H}^{(0)}-N^{k-1}H_0/(k-1)!=O(N^{k-2+20k(\log\log N)^{-1}}).$$

The desired formula for $\mathscr{H}^{(0)}$ follows from this and Lemma 5. $\mathscr{H}^{(i)}$ can be treated similarly.

Lemma 7. *As $N\to\infty$, we have*

$$F=\{(k-1)!t_1!\cdots t_k!\}^{-1}N^{k-1}\{y(N)\}^{t_1+\cdots+t_k}e^{-ky(N)}\{1+o(1)\}$$

uniformly in t_i with $t_i<2y(N)$.

The lemma follows from Lemmas 2 and 6.

Lemma 8. *Let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$. Let t_i $(i=1,\cdots,k)$ be positive integers such that $t_i=y(N)+x_i\sqrt{y(N)}$ with $\alpha_i<x_i<\beta_i$. Then, as $N\to\infty$, we have*

$$F=\{(k-1)!\}^{-1}N^{k-1}\{2\pi y(N)\}^{-k/2}e^{-(x_1^2+\cdots+x_k^2)/2}\{1+o(1)\}$$

uniformly in t_i.

This Lemma corresponds to Lemma 6 in [1]. The Stirling formula is used in the proof.

Lemma 9. *Let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$. Let $A^*(N)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$y(N)+\alpha_i\sqrt{y(N)}<\omega_N(n_i)<y(N)+\beta_i\sqrt{y(N)}\qquad(i=1,\cdots,k)$$

simultaneously. Then, as $N\to\infty$, we have

$$A^*(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-k/2}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-x^2/2}dx.$$

This lemma corresponds to Lemma 7 in [1]. It can also be proved by (1) that

$$\sum_{n_1+\cdots+n_k=N}\{\omega(n_i)-\omega_N(n_i)\}=O(N^{k-1}\log\log\log N).$$

The theorem now follows from this, (1), and Lemma 9 by similar way as the proofs of Lemmas 8 and 9 in [1].

We could prove the theorem by induction on k. By this, however, the proof will not essentially be shortened.

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References

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