98. On Discontinuous Groups Acting on a Real Hyperbolic Space. I

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1. This note gives a necessary and sufficient condition for a polyhedron in an n-dimensional real hyperbolic space to be a fundamental domain of some discontinuous group which has been established by B. Maskit [1] in 3-dimensional case. In 3-dimensional case our condition improves his parabolicity one and is much more combinatorial (See Definition 2). It also gives informations as to fixed point groups (See Theorem 2) analogous to the case of Coxeter groups ([2] Chapter IV).

We fix our notations. Let $\boldsymbol{H}$ be the subspace $\{\xi \in \boldsymbol{R}^n \mid \xi_1^2 + \cdots + \xi_n^2 < 1\}$ of $\boldsymbol{R}^n$ with the metric form $ds^2 = 4\sum_{j=1}^n d\xi_j^2/(1-\sum_{j=1}^n \xi_j^2)^2$ $(n \geqq 2)$, which is called an n-dimensional real hyperbolic space. Let G be the group of all isometries of $\boldsymbol{H}$. Let F be an n-dimensional open polyhedron with totally geodesic faces in $\boldsymbol{H}$ satisfying the conditions: i) the number of faces is finite, ii) $\overline{\overline{F}} \cap \partial \boldsymbol{H}$ is a finite set, where $\overline{\overline{F}}$ means the closure of F and $\partial \boldsymbol{H}$ the boundary of $\boldsymbol{H}$ both under the topology of $\boldsymbol{R}^n$.

Some concrete examples will be given in the part II.

2. We define two kinds of "fitness" for F as follows.

Definition 1. A discrete subgroup Γ of G is said to be *fit* for F if the following conditions are satisfied: i) $\bigcup_{\gamma \in \Gamma} \gamma\overline{F} = \boldsymbol{H}$ where $\overline{F}$ means the closure of F under the topology of $\boldsymbol{H}$; for any element γ in Γ which is not the unit element we have $F \cap \gamma F = \emptyset$ and the family of the subsets $\{\gamma\overline{F}\}_{\gamma \in \Gamma}$ in $\boldsymbol{H}$ is locally finite, ii) Γ has no reflection and iii) the subset $\{\gamma \in \Gamma \mid \overline{F} \cap \gamma\overline{F} = \emptyset\}$ of Γ consists of finite elements.

Definition 2. Let $\mathcal{P}$ be a subdivision of $\overline{F}$ and $\mathcal{A} = \{\gamma_1, \cdots, \gamma_a\}$ be a subset of G. A pair $(\mathcal{P}, \mathcal{A})$ is said to be *fit for* F if the following conditions are satisfied:

i) (*Structure of cell complex for* $\overline{F}$). $\mathcal{P}$ consists of a finite number of polyhedra each of which, called *facet*, is open in its support, namely the minimal subspace of $\boldsymbol{H}$ containing this facet. F is an element of $\mathcal{P}$. For any $F' \in \mathcal{P}$, $\overline{F'}$ is equal to the sum of $F'' \in \mathcal{P}$ such that $F'' \subset \overline{F'}$.

ii) (*Compatibility condition for* $\mathcal{P}$ *and* $\mathcal{A}$). Let $\mathcal{P}^{(\nu)}$ be the set of all $F' \in \mathcal{P}$ such that F' is ν-dimensional $(0 \leqq \nu \leqq n)$. First $\mathcal{P}^{(n-1)}$ consists of even number of faces $\{H_1^+, H_1^-, \cdots, H_a^+, H_a^-\}$ such that $\gamma_i H_i^+ = H_i^-$ and $\gamma_i F \cap F = \emptyset$ $(1 \leqq i \leqq a)$. Secondly for any $F' \in \mathcal{P}$ we have $\gamma_i F' \in \mathcal{P}$ when ever $F' \subset H_i^+$. We say that a facet F' is *linked with* F'' by γ_i^ε if

$F' \subset \overline{H_i^{-\varepsilon}}$ and $F' = \gamma_i^{\varepsilon} F''$ where ε means the signs ± 1 and $(H_i^{\varepsilon}, \gamma_i^{\varepsilon})$ means (H_i^+, γ_i) or (H_i^-, γ_i^{-1}) according as $\varepsilon = 1$ or $\varepsilon = -1$. We also say that F' and F'', both in $\mathcal{P}$, are *equivalent* if there exists a sequence of facets $F_1', \cdots, F_k'$ such that $F' = F_1', F'' = F_k'$ and that F_j' is linked with F_{j+1}' by some $\alpha_j \in \mathcal{A} \cup \mathcal{A}^{-1}$ $(1 \leqq j \leqq k)$. So F'' is transformed to F' by the product $\alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \in G$ which will be called '*gallery from F'' to F'*'. Let $F' = F_1', F_2', \cdots, F_{\mu(F')}'$ be all the elements in $\mathcal{P}$ equivalent to F' and $H_{i(F',j,k)}^{\varepsilon(F',j,k)}$ $(1 \leqq j \leqq \mu(F'), 1 \leqq k \leqq \nu(F', j))$ be all the elements in $\mathcal{P}^{(n-1)}$ such that $\overline{H_{i(F',j,k)}^{\varepsilon(F',j,k)}} \supset F_j'$. Choose galleries $\tau_{F',j}$ $(1 \leqq j \leqq \mu(F'))$ from F_j' to F'. Let $g_{F',j,k}$ be the product $\tau_{F',l(j)} \gamma_{i(j,k)}^{\varepsilon(j,k)} \tau_{F',j}^{-1}$ where $l(j)$ denotes the number such that $\gamma_{i(j,k)}^{\varepsilon(j,k)} F_j' = F_{l(j)}'$. $g_{F',j,k}$ is a gallery from F' to itself. The last condition is that each $g_{F',j,k}$ $(1 \leqq j \leqq \mu(F'), 1 \leqq k \leqq \nu(F', j))$ fixes F' pointwise when F' runs through representatives of $\mathcal{P}$ under the equivalence relation.

iii) (*Unramifiedness condition for $\mathcal{P}^{(n-2)}$*). Let $\mathcal{P}^{(n-2)} = \bigcup_{r=1}^{b} \mathcal{P}_r^{(n-2)}$ be the partition of $\mathcal{P}^{(n-2)}$ under the equivalence relation introduced above. Then there exists a positive integer m_r corresponding to each equivalence class $\mathcal{P}_r^{(n-2)}$ such that

1) $$\sum_{N \in \mathcal{P}_r^{(n-2)}} \angle N = 2\pi / m_r$$

where $\angle N$ means the angle between the two $(n-1)$-dimensional facets $H_i^{\varepsilon}, H_{i'}^{\varepsilon'}$ such that $\overline{H_i^{\varepsilon}} \cap \overline{H_{i'}^{\varepsilon'}} \supset N$. Let $\mathcal{P}_r^{(n-2)}$ consist of $N_1, \cdots, N_s$ such that $N_j \subset \overline{H_{i_j}^{-\varepsilon_j}}$, $N_j = \gamma_{i_j}^{\varepsilon_j} N_{j+1}$ $(1 \leqq j \leqq s)$ where we put $N_{s+1} = N_1$. We take $\gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_{j-1}}^{\varepsilon_{j-1}}$ as $\tau_{N,j}$ $(1 \leqq j \leqq s)$. Then $g_{N,j,k}$ is equal to $g_{N_1} = \gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_s}^{\varepsilon_s}$, $g_{N_1}^{-1}$ or the identity transformation. From the condition ii), iii) the following relation holds;

2) $$\mathcal{R}_r : g_{N_1}^{m_r} = \text{the identity}$$

iv) (*Parabolicity condition for $\overline{\overline{F}} \cap \partial \boldsymbol{H}$*). The link and the equivalence relation above also apply to the set $\overline{\overline{F}} \cap \partial \boldsymbol{H}$ replacing $\overline{H_i^{\varepsilon}}$ by $\overline{\overline{H_i^{\varepsilon}}}$ and F' by c where c runs over the representatives of $\overline{\overline{F}} \cap \partial \boldsymbol{H}$ under the equivalence relation. Then g_c $(1 \leqq j \leqq \mu(c), 1 \leqq k \leqq \nu(c, j))$ are all parabolic.

3. Main theorems

Theorem 1. *For any discrete subgroup Γ fit for F there exists a pair $(\mathcal{P}, \mathcal{A})$ fit for F such that $\langle \mathcal{A} \rangle = \Gamma$ where $\langle \mathcal{A} \rangle$ means a group generated by $\mathcal{A}$ in G. Conversely for any pair $(\mathcal{P}, \mathcal{A})$ fit for F the group $\langle \mathcal{A} \rangle$ is fit for F.*

Corollary. *Let $(\mathcal{P}, \mathcal{A})$ be fit for F. Γ be the group $\langle \mathcal{A} \rangle$. Then $\{\tau F' \mid \tau \in \Gamma, F' \in \mathcal{P}\}$, considered as a set of subset of $\boldsymbol{H}$, gives a subdivision $\mathcal{P}_{\boldsymbol{H}}$ of $\boldsymbol{H}$. $\mathcal{P}_{\boldsymbol{H}}$ is locally finite and has a structure of cell complex naturally induced from that of $\mathcal{P}$. The representatives of $\mathcal{P}_{\boldsymbol{H}}$ under the action of Γ are the representatives of $\mathcal{P}$ under the equivalence re-*

ation defined in Definition 2–ii). *Let* $F' \in \mathcal{P}$. *Then for any* $x \in F'$ *the isotropy subgroup* Γ_x *are the same which we denote by* $\Gamma_{F'}$.

From now on for covenience we assume the empty facet $\emptyset$ belongs to $\mathcal{P}$ and denote by $\Gamma_\emptyset$ the whole group Γ.

Theorem 2. *Let* $(\mathcal{P}, \mathcal{A})$ *be fit for* F *and* F' *an element of* $\mathcal{P}$ *or* $\overline{\overline{F}} \cap \partial \boldsymbol{H}$. *Then* $\Gamma_{F'}$ *is generated by* $g_{F',j,k}$ $(1 \leqq j \leqq \mu(F'), 1 \leqq k \leqq \nu(F', j))$ (see Definition 2). *Let* $S_{F'}$ *be a sufficiently small sphere with centre* $x \in F'$ *and normal to the support of* F', *or a small cuspidal sphere at* $F'-\{F'\}$ *according as* $F' \in \mathcal{P}$ *or* $F' \in \overline{\overline{F}} \cap \partial \boldsymbol{H}$ *so that* $\tau_{F',j} F \cap S_{F'}$ *gives a "vertex figure" of* $\tau_{F',j} F$ *at* F'. *When* F' *is empty we mean by* $S_{F'}$ *the total space* $\boldsymbol{H}$. *Then* $\Gamma_{F'}$ *acts on* $S_{F'}$ *whose fundamental domain is given by* $\bigcup_{j=1}^{\mu(F')} \tau_{F',j} \overline{F} \cap S_{F'}$. *By a suitable selection of* $\tau_{F',j}$, *we can make this set and its interior both connected. The latter will be denoted by* $F_{F'}$. *Then the set of those* $\tau_{F',j} H_{i(F',j,k)}^{\varepsilon(F',j,k)} \cap S_{F'}$ *which are on the boundary of* $F_{F'}$ *consists of an even number of geodesic polyhedra with codimension* 1 *in* $S_{F'}$, F', $H_{F',i}^{+}$, $H_{F',i}^{-}$ $(1 \leqq i \leqq a(F'))$, *such that* $g_{F',j,k} H_{F',i}^{+} = H_{F',i}^{-}$ *for some* $g_{F',j,k}$ *which we denote by* $\gamma_{F',i}$. *The set* $\mathcal{A}_{F'}$ *of* $\gamma_{F',i}$ $(1 \leqq i \leqq a(F'))$ *also generates* $\Gamma_{F'}$. *For each* $\mathcal{P}_r^{(n-2)}$ *such that* $\overline{N} \supset F'_j$ *for some* $N \ni \mathcal{P}_r^{(n-2)}$ *and* F'_j *equivalent to* F', *we have a relation of the following form*

3) $\mathcal{R}_{F',r}: ((\tau_{F',j_1} \gamma_{i_1}^{\varepsilon_1} \tau_{F',j_2}^{-1}) \cdot (\tau_{F',j_2} \gamma_{i_2}^{\varepsilon_2} \tau_{F',j_3}^{-1}) \cdots (\tau_{F',j_s} \gamma_{i_s}^{\varepsilon_s} \tau_{F',j_1}^{-1}))^{m_1} =$ *the identity*, *determining the indices* j_λ *by the condition that* F'_{j_λ} *is linked with* $F'_{j_{\lambda+1}}$, N_λ *with* $N_{\lambda+1}$, *both by* $\gamma_{i_\lambda}^{\varepsilon_\lambda}$. *This relation is reduced to* $\mathcal{R}_r$ (See Definition 2–iii)) *in the total group* Γ, *and each term* $\tau_{F',j_\lambda} \gamma_{i_\lambda}^{\varepsilon_\lambda} \tau_{F',j_{\lambda+1}}^{-1}$ *is equal to one of* $\gamma_{F',i}^{\varepsilon}$ *or simply the identity which can be neglected. Let* $\mathcal{R}_{F'}$ *be the set of all such relations* $\mathcal{R}_{F',r}$ *associated with* F'. *Then* $\mathcal{R}_{F'}$ *defines a system of fundamental relations in the generator* $\mathcal{A}_{F'}$ *of the group* $\Gamma_{F'}$.

4. Proof. Using elementary combinatorial arguments the existence of a pair $(\mathcal{P}, \mathcal{A})$ fit for F turns out to be necessary for a discrete subgroup Γ of G to be fit for F. From the converse part of Theorem 1, Theorem 2 immediately follows applying a general result of A. M. Macbeath [3] to the group $\Gamma_{F'}$ acting on the simply connected space $S_{F'}$. To prove the converse part of Theorem 1, we construct a space $\mathcal{Z}$ as follows. Let $(\mathcal{P}, \mathcal{A})$ be fit for F and Γ be the group $\langle \mathcal{A} \rangle$. For each $\tau \in \Gamma$ we consider $\tau \overline{F}$ to be a topological space with the induced topology of $\boldsymbol{H}$ and denote it by $\tilde{q}(\tau)$. Let $\mathcal{Z}_0$ be $\coprod_{\tau \in \Gamma} \tilde{q}(\tau)$ (disjoint union). Pasting each $\tilde{q}(\tau)$ and each $\tilde{q}(\tau \gamma_i^{\varepsilon})$ along the common face $\tau \overline{H_i^{-\varepsilon}} = \tau \gamma_i^{\varepsilon} \overline{H_i^{\varepsilon}}$ we get an identification space $\mathcal{Z}$. Γ acts on $\mathcal{Z}$ in a natural manner and also we have an Γ^{-} equivariant projection map ψ from $\mathcal{Z}$ into $\boldsymbol{H}$. Then the problem is reduced to show that $\psi: \mathcal{Z} \to \boldsymbol{H}$ is a homeomorphism. This follows from the following Lemma using the fitness condition.

Lemma. *Let X, Y be topological spaces and $\psi: X \to Y$ be a map such that* i) *X is a Hausdorff arcwise connected space,* ii) *Y is a connected metric space with a distance function d and* iii) *there exists a real number $d_0 > 0$ satisfying the following condition: for any $x \in X$ we can find an open neighbourhood O_x of x such that $\psi | O_x : O_x \to V(\psi(x), d_0)$ is a homeomorphism where $\psi | O_x$ is the restriction of ψ on O_x and $V(\psi(x), d_0)$ means $\{y \in Y \mid d(y, \psi(x)) < d_0\}$. Also $V(\psi(x), d_1)$ is connected whenever $O < d_1 \leqq d_0$. Then ψ is a covering map from X onto Y. In particular if Y is simply connected then $\psi: X \to Y$ is a homeomorphism.*

The proof is easy.

5. **Remark.** By a slite modification of our formulations the condition ii) of Definition 1 can be removed. It is an open problem to extend the result to any irreducible non-compact symmetric space.

References

[1] B. Maskit: On Poincaré's theorem for fundamental polygons. Advances in Math., **7**, 219–230 (1971).

[2] N. Bourbaki: Groupes et algèbres de Lie, Chapitres 4, 5 et 6. Eléments de Mathematique, XXXIV. Paris, Hermann (1968).

[3] A. M. Macbeath: Groups of homeomorphisms of a simply connected space. Ann. of Math., **79**, 473–487 (1964).