115. A Note on the Classification of Stability

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1. Introduction. We shall consider the system of ordinary differential equations $\dot{x}=f(t,x)$. Let R^n denote Euclidean space of dimension n. We shall assume that f is continuous on $[0,\infty)\times R^n$ and satisfies the equality $f(t,0)=0$ for $t\geqq 0$. In this note we discuss various types of stability, that is, (simple) stability (abbreviated by S), uniform stability (US), quasi-asymptotic stability (in the large) ($QAS(L)$), stability (in the large) ($AS(L)$), quasi-equi-asymptotic stability (in the large) ($QEAS(L)$), equi-asymptotic stability (in the large) ($EAS(L)$), quasi-uniform-asymptotic stability (in the large) ($QUAS(L)$), uniform-asymptotic stability (in the large) ($UAS(L)$), and exponential-asymptotic stability (in the large) ($Exp\ AS(L)$) introduced by Lyapunov, Massera and many others. For the definitions of the above notions we shall employ those in Yoshizawa [4].

Our purpose is to clarify the relations between these notions. This note is based on a portion of a dissertation of the author's Master degree in 1975 submitted to Osaka University.

We now define $F(S)$ as the family of continuous functions f for which the trivial solutions $x(t)=0$ of $\dot{x}=f(t,x)$ are stable. Of course, in a way similar to the above notation we also define $F(US), F(QAS), \cdots, F(Exp\ ASL)$ respectively. It is convenient to define

$$F_{Lin}(*)=\{f\in F(*)\mid f(t,x)=A(t)x\},$$
$$F_{Aut}(*)=\{f\in F(*)\mid f \text{ is independent of } t\}$$
and $$F_{Per}(*)=\{f\in F(*)\mid f(t+\omega,x)=f(t,x) \text{ for some } \omega>0\}.$$

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2. Well-known Relations. First we begin to give the propositions, immediate consequences of the definitions.

Proposition 1. $F(US)\subset F(S)$.

Proposition 2. (i) $F(QUAS)\subset F(QEAS)\subset F(QAS)$,
(ii) $F(Exp\ AS)\subset F(UAS)\subset F(EAS)\subset F(AS)$.

Proposition 3. (i) $F(QUASL)\subset F(QEASL)\subset F(QASL)$,
(ii) $F(Exp\ ASL)\subset F(UASL)\subset F(EASL)\subset F(ASL)$.

Proposition 4. (i) $F(Exp\ ASL)\subset F(Exp\ AS)$,
(ii) $F(QUASL)\subset F(QUAS)$ hence $F(UASL)\subset F(UAS)$,
(iii) $F(QEASL)\subset F(QEAS)$ hence $F(EASL)\subset F(EAS)$,

(iv) $F(QASL)\subset F(QAS)$ hence $F(ASL)\subset F(AS)$.

Next we give the following relations which show that for certain important classes of systems of differential equations, some concepts are equivalent.

Theorem 1. *For the class of linear systems,*

(i) $F_{Lin}(EASL)=F_{Lin}(ASL)=F_{Lin}(EAS)=F_{Lin}(AS)$
$=F_{Lin}(QEASL)=F_{Lin}(QASL)=F_{Lin}(QEAS)=F_{Lin}(QAS)$,

(ii) $F_{Lin}(QUASL)=F_{Lin}(QUAS)$,

(iii) $F_{Lin}(Exp\ ASL)=F_{Lin}(UASL)=F_{Lin}(Exp\ AS)=F_{Lin}(USA)$.

Theorem 2. *For the class of periodic systems,*

(i) $F_{Per}(US)=F_{Per}(S)$,

(ii) $F_{Per}(UAS)=F_{Per}(EAS)=F_{Per}(AS)$,

(iii) $F_{Per}(UASL)=F_{Per}(EASL)=F_{Per}(ASL)$.

Theorem 3. *For the class of autonomous systems,*

(i) $F_{Aut}(US)=F_{Aut}(S)$,

(ii) $F_{Aut}(UAS)=F_{Aut}(EAS)=F_{Aut}(AS)$,

(iii) $F_{Aut}(UASL)=F_{Aut}(EASL)=F_{Aut}(ASL)$.

These theorems are well-known. See Yoshizawa [4].

3. Examples. Each of types of asymptotic stability in the large is the property concerning the behaviors of solutions in the whole space $[0,\infty)\times R^n$, while each of types of asymptotic stability is that only in some neighborhood of the trivial solution. This shows, in general, that each of types of asymptotic stability does not imply any type of asymptotic stability in the large. The following examples illustrate that arbitrary two types of stability are actually different concepts. We mean that x, x_1, x_2 and y in the following examples are real numbers.

Example 1. $F_{Aut}(UASL)\subsetneq F_{Aut}(Exp\ AS)$, hence $F(UASL)\subsetneq F(Exp\ AS)$. Consider the equation $\dot{x}=f(x), f(x)=-x^3$. Then $x(t:t_0,x_0)=\dfrac{x_0}{\sqrt{2(t-t_0)+1}}$ is the solution of this equation through (t_0,x_0). Therefore we can see that $f\in F_{Aut}(UASL)$, but $f\notin F_{Aut}(Exp\ AS)$.

Example 2. $F_{Lin}(EASL)\subsetneq F_{Lin}(UAS)$, hence $F(EASL)\subsetneq F(UAS)$. Consider the equation $\dot{x}=f(t,x), f(t,x)=-\dfrac{1}{t+1}x$. For $x_0\neq 0$, $x(t:t_0,x_0)=\dfrac{t_0+1}{t+1}x_0$ satisfies that $x(2m-1:m-1,x_0)=\dfrac{1}{2}x_0\not\to 0$ as $m\to\infty$, hence $f\notin F_{Lin}(QUAS)$. On the other hand we can see that $f\in F_{Lin}(EASL)\cap F_{Lin}(US)$.

Example 3. $F_{Lin}(EASL)\subsetneq F_{Lin}(US)$. Consider the equation, given by Massera [2], $\dot{x}=f(t,x)$ where

$$f(t,x)=-\{13+12\sin\log(t+1)+12t(t+1)^{-1}\cos\log(t+1)\}\cdot x.$$

It is easily verified that $f\in F_{Lin}(EASL)$, but $f\notin F_{Lin}(QUAS)\cup F_{Lin}(US)$.

Example 4. $F(ASL) \subsetneq F(EAS)$. Consider the system of equations $\dot{x}=\dfrac{(\partial/\partial t)g(t,y)-y(\partial/\partial y)g(t,y)}{g(t,y)}\cdot x$, $\dot{y}=-y$ where

$$g(t,y)=\frac{\sin^4(ye^t)}{\sin^4(ye^t)+\{1-t\sin^2(ye^t)\}^2}+\frac{1}{1+\sin^4(ye^t)}\cdot\frac{1}{1+t^2}.$$

Refer to Massera [1]. We can easily show that the trivial solution $x(t)=0$, $y(t)=0$ is asymptotically stable in the large, but is not equi-asymptotically stable.

Example 5. $F_{Lin}(QUASL) \subsetneq F_{Lin}(US)$, hence $F_{Lin}(QUASL) \subsetneq F_{Lin}(UAS)$. Consider the equation, given by Massera [2], $\dot{x}=f(t,x)$, $f(t,x)=(6t\sin t-2t)x$. Then we have that $f\in F_{Lin}(QUASL)$, but $f\notin F_{Lin}(US)$.

Remark. We can easily see that the trivial solution is (simple) stable, if it is unique to the right on $[0,\infty)$ and quasi-equi-asymptotically stable. For example, if we assume that f satisfies locally a Lipschitz condition with respect to x, then for the system of equations $\dot{x}=f(t,x)$, quasi-equi-asymptotic stability implies (simple) stability, hence it implies equi-asymptotic stability.

Example 6. $F_{Aut}(QASL) \subsetneq F_{Aut}(QEAS)\cup F_{Aut}(S)$, hence $F(QASL) \subsetneq F(QEAS)\cup F(S)$. Consider the system of equation $\dfrac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=f(x_1,x_2)$ where $f(x_1,x_2)=\begin{pmatrix}x_1^2(x_2-x_1)+x_2^5\\x_2^2(x_2-2x_1)\end{pmatrix}$. Refer to Vinograd's example in Hahn [5], 191~194. An elementary calculation verifies that $f\in F_{Aut}(QASL)$, but $f\notin F_{Aut}(QEAS)\cup F_{Aut}(S)$.

Example 7. $F_{Aut}(UAS) \subsetneq F_{Aut}(Exp\,AS)$, hence $F(UAS) \subsetneq F(\mathrm{Exp}\,AS)$. Consider the equation $\dot{x}=f(x)$, $f(x)=x^4-2x^3$. Then, for $0<x<1$ we have $\dot{x}<-x^3$ and for $-1<x<0$ we have $\dot{x}>-x^3$, while we have $\dot{x}<-3x^3$ for $-1<x<0$. A comparison theorem shows that $f\in F_{Aut}(UAS)$, but $f\notin F_{Aut}(\mathrm{Exp}\,AS)$.

Example 8. $F(EAS) \subsetneq F(UAS)$. Consider the equation $\dot{x}=f(t,x)$, $f(t,x)=\dfrac{1}{t+1}(x^2-x)$. Then $x(t:t_0,x_0)=\dfrac{x_0(t_0+1)}{x_0(t_0+1)-(x_0-1)(t+1)}$ is the solution of this equation through (t_0,x_0). It is obvious that $f\in F(EAS)$. However, for $0<x_0<1$ it follows that

$$x(2m-1:m-1,x_0)=\frac{x_0}{2-x_0}\not\to 0 \qquad \text{as } m\to\infty.$$

Thus $f\notin F(QUAS)$, hence $f\notin F(UAS)$.

Example 9. $F(AS)\not\subset F(EAS)$. Consider the system of equations

$$\dot{x}=\frac{(\partial/\partial t)g(t,y)+y(y-1)(\partial/\partial y)g(t,y)}{g(t,y)}\cdot x,\ \dot{y}=y(y-1)$$

where

$$g(t, y)=\frac{\sin^4(ye^t)}{\sin^4(ye^t)+\{1-t\sin^2(ye^t)\}^2}+\frac{1}{1+\sin^4(ye^t)}\cdot\frac{1}{1+t^2}.$$

It is easily shown that the trivial solution is asymptotically stable, but is not equi-asymptotically stable.

Example 10. $F_{Lin}(US) \subsetneq F_{Lin}(AS)$, hence $F(US) \subsetneq F(AS)$. Consider the equation $\dot{x}=f(t,x)$, $f(t,x)=x\cos t$. For $t \geqq t_0$, the solution $x(t: t_0, x_0)=x_0 \exp[\sin t-\sin t_0]$ satisfies the inequalities $|x_0|e^{-2} \leqq |x(t: t_0, x_0)| \leqq |x_0|e^2$. Hence $f \in F_{Lin}(US)$, but $f \notin F_{Lin}(AS)$.

Example 11. $F_{Aut}(US) \subsetneq F_{Aut}(AS)$. Consider the equation $\dot{x}=f(x)$, $f(x)=\begin{cases} x^3 \sin\dfrac{1}{x}, & x \neq 0 \\ 0 & , x=0 \end{cases}$. It is obvious that each of the solutions $x(t)=0$, $x(t)=\dfrac{1}{k\pi}$ (k: nonzero integer) is unique to the right on $[0, \infty)$. Therefore, we can see that $f \in F_{Aut}(US)$, but $f \notin F_{Aut}(AS)$.

Example 12. $F_{Lin}(S) \subsetneq F_{Lin}(US)$, hence $F(S) \subsetneq F(US)$. Consider the equation $\dot{x}=f(t,x)$, $f(t,x)=(t\sin t-1)x$. Then we obtain the solution

$$x(t: t_0, x_0)=x_0 \exp[-t-t\cos t+\sin t+t_0+t_0\cos t_0-\sin t_0].$$

It is clearly valid that $f \in F_{Lin}(S)$. However $f \notin F_{Lin}(US)$, because for $x_0 \neq 0$, it follows that

$$|x((2m+1)\pi: 2m\pi, x_0)|=|x_0|\, e^{4m\pi} \to \infty \qquad \text{as } m \to \infty.$$

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