# 128. *Canonical Forms of Some Systems of Linear Partial Differential Equations*

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**§1. Introduction.** It is well known that the linear ordinary differential equation of the second order

$$\frac{d^2u}{dx^2}=q_1(x)\frac{du}{dx}+q_2(x)u \tag{1}$$

is transformed by a change of variable

$$u=a(x)v$$

into an equation of canonical form

$$\frac{d^2v}{dx^2}=p(x)v \tag{2}$$

and that the coefficient $p(x)$ is equal to the Schwarzian derivative of the quotient of two linearly independent solutions of (1). Note that the equation of canonical form (2) is characterized by the condition that the Wronskian of two solutions is reduced to a constant.

In this paper, we shall extend these facts to two kinds of completely integrable systems of partial differential equations:

$$\frac{\partial^2 u}{\partial x_j\partial x_k}=\sum_{\alpha=1}^{n} q_{jk}^{\alpha}(x)\frac{\partial u}{\partial x_\alpha}+q_{jk}^0(x)u \qquad (1\leq j,\ k\leq n) \tag{E}$$

and

$$\begin{cases}\dfrac{\partial u}{\partial x_j}=a_j(x)\dfrac{\partial u}{\partial x_j}+b_j(x)u & (1\leq j\leq n)\\[2mm] \dfrac{\partial^2 u}{\partial x_j\partial x_k}=c_{jk}(x)\dfrac{\partial u}{\partial x_{i_0}}+d_{jk}(x)u & (1\leq j,\ k\leq n)\end{cases} \tag{$E_{i_0}$}$$

where all coefficients are supposed to be holomorphic in a domain of $\boldsymbol{C}^n$ and $a_{i_0}(x)\equiv 1$, $b_{i_0}(x)\equiv 0$.

**§2. Canonical form of system (*E*).** Given $n+1$ functions $u_0, u_1, \cdots, u_n$ holomorphic in $x_1, \cdots, x_n$, we call

$$\det\begin{bmatrix} u_0 & \cdots & u_n \\ \dfrac{\partial u_0}{\partial x_1} & & \dfrac{\partial u_n}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial u_0}{\partial x_n} & & \dfrac{\partial u_n}{\partial x_n}\end{bmatrix}$$

the Wronskian of $u_0, u_1, \cdots, u_n$ and denote by $W(u_0, u_1, \cdots, u_n)$.

In the same way as for the ordinary differential equation (1), we have

**Proposition 1.** *For $n+1$ solutions $u_0, u_1, \cdots, u_n$ of system $(E)$, we have*

$$dW(u_0, u_1, \cdots, u_n)=\Big(\sum_j \sum_l q^l_{jl}(x)dx_j\Big)W(u_0, u_1, \cdots, u_n).$$

We say that system $(E)$ is of canonical form if the Wronskian of $n+1$ solutions of system $(E)$ is reduced to a constant. The following proposition is an immediate consequence of Proposition 1.

**Proposition 2.** *System $(E)$ is of canonical form if and only if*

$$\sum_l q^l_{jl}(x)=0 \qquad (1\leq j\leq n).$$

We obtain from Proposition 2 the following

**Proposition 3.** *System $(E)$ is transformed by a change of variable*

$$u=a(x)v$$

*into a system of canonical form*

$$(E^0) \qquad \frac{\partial^2 v}{\partial x_j \partial x_k}=\sum_\alpha p^\alpha_{jk}(x)\frac{\partial v}{\partial x_\alpha}+p^0_{jk}(x)v,$$

*where $a(x)$ satisfies*

$$da(x)=\frac{1}{n+1}\Big(\sum_j \sum_l q^l_{jl}(x)\,dx_j\Big)a(x)$$

*and $p^\alpha_{jk}(x)$ and $p^0_{jk}(x)$ are given by*

$$p^\alpha_{jk}=q^\alpha_{jk}-\frac{\delta^\alpha_j}{n+1}\sum_l q^l_{kl}-\frac{\delta^\alpha_k}{n+1}\sum_l q^l_{jl},$$

$$p^0_{jk}=q^0_{jk}-\frac{1}{n+1}\frac{\partial}{\partial x_j}\sum_l q^l_{kl}+\frac{1}{n+1}\sum_\alpha\sum_l q^l_{\alpha l}q^\alpha_{jk}-\frac{1}{(n+1)^2}\sum_l q^l_{jl}\cdot\sum_l q^l_{kl}.$$

*$\delta$ being the Kronecker symbol.*

In his thesis of master degree, T. Oda gave a definition of Schwarzian derivatives for a locally biholomorphic map $f$ of a domain of $C^n$ into $C^n$ as follows:

$$S^\alpha_{jk}(f)=\sum_l \frac{\partial^2 f_l}{\partial x_j \partial x_k}\frac{\partial x_\alpha}{\partial f_l}-\frac{1}{n+1}\Big\{\delta^\alpha_j \sum_{l,m}\frac{\partial^2 f_l}{\partial x_m \partial x_k}\frac{\partial x_m}{\partial f_l}+\delta^\alpha_k \sum_{l,m}\frac{\partial^2 f_l}{\partial x_m \partial x_j}\frac{\partial x_m}{\partial f_l}\Big\},$$

$$S^0_{jk}(f)=|\partial f/\partial x|^{1/(n+1)}\Big\{\frac{\partial^2}{\partial x_j \partial x_k}|\partial f/\partial x|^{-1/(n+1)}-\sum_\alpha \frac{\partial}{\partial x_\alpha}|\partial f/\partial x|^{-1/(n+1)}S^\alpha_{jk}(f)\Big\}.$$

**Proposition 4.** *Let $u_0, u_1, \cdots, u_n$ be a fundamental system of solutions of system $(E)$ and put $f=(u_1/u_0, \cdots, u_n/u_0)$. Then the coefficients $p^\alpha_{jk}(x)$ and $p^0_{jk}(x)$ of the canonical system $(E^0)$ associated with $(E)$ coincide with $S^\alpha_{jk}(f)$ and $S^0_{jk}(f)$ respectively.*

One should remark that the integrability condition yields relation among $p^\alpha_{jk}$, $p^0_{jk}$ and their derivatives and in particular that, if $n\geq 2$, $p^0_{jk}$ are expressible in terms of $p^\alpha_{jk}$ and their derivatives.

**§ 3. Canonical forms of system $(E_{i_0})$.** Given two functions $u_0, u_1$ holomorphic in $x_1, \cdots, x_n$, we call

$$\det\begin{pmatrix} u_0 & u_1 \\ \dfrac{\partial u_0}{\partial x_{l_0}} & \dfrac{\partial u_1}{\partial x_{l_0}} \end{pmatrix}$$

the $l_0$-Wronskian of $u_0$ and $u_1$ and denote by $W_{l_0}(u_0, u_1)$. We say that system $(E_{i_0})$ is of $l_0$-canonical form if the $l_0$-Wronskian of two solutions of system $(E_{i_0})$ is reduced to a constant. Corresponding to Propositions 1, 2 and 3, we have the following three propositions.

**Proposition 5.** *If $a_{l_0}(x)\not\equiv 0$, then for two solutions $u_0$ and $u_1$ of system $(E_{i_0})$, we have*

$$dW_{l_0}(u_0, u_1)=\sum_j g_j(x)dx_j W_{l_0}(u_0, u_1),$$

*where*

$$2g_j(x)=\frac{1}{a_{l_0}(x)}\frac{\partial a_{l_0}(x)}{\partial x_j}+b_j(x)+c_{ji_0}(x).$$

**Proposition 6.** *System $(E_{i_0})$ is of $l_0$-canonical form if and only if $a_{l_0}(x)\not\equiv 0$ and*

$$g_j(x)=0 \qquad (1\leq j\leq n).$$

**Proposition 7.** *If $a_{l_0}(x)\not\equiv 0$, system $(E_{i_0})$ is transformed by a change of variable*

$$u=g(x)v$$

*into a system of $l_0$-canonical form*

$$(E^{l_0}_{i_0}) \qquad \begin{cases} \dfrac{\partial v}{\partial x_j}=\tilde{a}_j(x)\dfrac{\partial v}{\partial x_{i_0}}+\tilde{b}_j(x)v \\ \dfrac{\partial^2 v}{\partial x_j\partial x_k}=\tilde{c}_{jk}(x)\dfrac{\partial v}{\partial x_{i_0}}+\tilde{d}_{jk}(x)v \end{cases}$$

*where $g(x)$ satisfies*

$$dg(x)=\frac{1}{2}\sum_j g_j(x)dx_j\cdot g(x)$$

*and the coefficients of $(E^{l_0}_{i_0})$ are given by*

$$\begin{aligned}
&\tilde{a}_j=a_j, \qquad \tilde{b}_j=b_j-g_j+a_jg_j, \\
&\tilde{c}_{jk}=c_{jk}-a_kg_j-a_jg_k, \\
&\tilde{d}_{jk}=d_{jk}-b_kg_j-b_jg_k+c_{jk}g_l-\frac{\partial}{\partial x_j}g_k-g_jg_k.
\end{aligned}$$

Using the following differential expressions for $f$ holomorphic in $x_1, \cdots, x_n$

$$D_j(f)=\frac{\partial f}{\partial x_j},$$

$$P_{ljk}(f)=\frac{\partial^2 f}{\partial x_l\partial x_j}\frac{\partial f}{\partial x_k}-\frac{\partial^2 f}{\partial x_k\partial x_j}\frac{\partial f}{\partial x_l},$$

$$S^{im}_{ljk}(f)=\frac{\partial^2 f}{\partial x_i \partial x_m}\{P_{ljk}(f)+P_{lkj}(f)\}$$
$$-\left\{3\frac{\partial^2 f}{\partial x_l \partial x_j}\frac{\partial^2 f}{\partial x_m \partial x_k}-2\frac{\partial^3 f}{\partial x_l \partial x_j \partial x_k}\frac{\partial f}{\partial x_m}\right\}\frac{\partial f}{\partial x_i},$$

we have the following proposition analogous to Proposition 4.

**Proposition 8.** *Let $u_0$ and $u_1$ be a fundamental system of solutions of system $(E_{i_0})$ and put $f=u_1/u_0$. Then the $l_0$-canonical system $(E^{l_0}_{i_0})$ associated with $(E_{i_0})$ is written as*

$$(3)\quad \begin{cases} \dfrac{\partial v}{\partial x_j}=\dfrac{D_j(f)}{D_{i_0}(f)}\dfrac{\partial v}{\partial x_{i_0}}-\dfrac{1}{2}\dfrac{P_{il_0j}(f)}{D_{i_0}(f)D_{l_0}(f)}v, \\ \dfrac{\partial v}{\partial x_j \partial x_k}=\dfrac{1}{2}\dfrac{P_{l_0jk}(f)+P_{l_0kj}(f)}{D_{i_0}(f)D_{l_0}(f)}\dfrac{\partial v}{x_{i_0}}-\dfrac{1}{4}\dfrac{S^{i_0l_0}_{l_0jk}(f)}{D_{i_0}(f)D_{l_0}(f)}v, \end{cases}$$

We say that differential expression $R(f)$ is a relative invariant of degree $r$ if

$$R\left(\frac{af+b}{cf+d}\right)=\frac{1}{(cf+d)^{2r}}R(f)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,C)$. A relative invariant of degree 0 is called a Schwarzian derivative.

**Proposition 9.** *The differential expression $D_j(f)$, $P_{ljk}(f)$ and $S^{im}_{ljk}(f)+S^{il}_{mjk}(f)$ are relative invariants of degree 1, 2 and 3 respectively and the coefficients of the system (3) are Schwarzian derivatives.*

**Proposition 10.** *Let $\bar{x}_1=(\bar{x}_1,\cdots,\bar{x}_u)$ be another system of coordinates and let $D_\alpha(\bar{f})$, $P_{\alpha\beta\gamma}(\bar{f})$, $S^{\mu\nu}_{\alpha\beta\gamma}(\bar{f})$ be the corresponding operation for $\bar{f}(\bar{x})=f(x)$ in terms of $\bar{x}$. Then we have the following formulas.*

$$D_j(f)=D_\alpha(\bar{f})J^\alpha_j,$$
$$P_{ljk}(f)=P_{\alpha\beta\gamma}(\bar{f})J^\alpha_l J^\beta_j J^\gamma_k+D_\alpha(\bar{f})D_\gamma(\bar{f})\mathcal{P}^{(\alpha,\gamma)}_{ljk},$$
$$S^{im}_{ljk}(f)+S^{il}_{mjk}(f)=(S^{\mu\nu}_{\alpha\beta\gamma}(\bar{f})+S^{\mu\alpha}_{\nu\beta\gamma}(\bar{f}))J^\alpha_l J^\beta_j J^\gamma_k J^\mu_i J^\nu_m$$
$$+D_\mu(\bar{f})P_{\alpha\beta\gamma}(\tilde{f})(Q^{(\mu\alpha\beta\gamma)}_{imljk}+Q^{(\mu\alpha\beta\gamma)}_{ilmjk})$$
$$+D_\mu(\bar{f})D_\alpha(\bar{f})D_\gamma(\bar{f})\left(\overset{(\mu\alpha\gamma)}{\mathcal{S}^{im}_{ljk}}+\overset{(\mu\alpha\gamma)}{\mathcal{S}^{il}_{mjk}}\right),$$

*where*

$$J^\alpha_i=\partial\bar{x}_\alpha/\partial x_i,\ J^\alpha_{il}=\partial J^\alpha_i/\partial x_l,\ J^\alpha_{ljk}=\partial J^\alpha_{lj}/\partial x_k,$$
$$\mathcal{P}^{(\alpha\gamma)}_{ljk}=J^\alpha_{lj}J^\gamma_k-J^\alpha_{kj}J^\gamma_l,$$
$$\overset{(\mu\alpha\gamma)}{\mathcal{S}^{im}_{ljk}}=J^\mu_{im}(\mathcal{P}^{(\alpha\gamma)}_{lkj}+\mathcal{P}^{(\alpha\gamma)}_{ljk})-J^\mu_i(3J^\alpha_{lj}J^\gamma_{mk}-2J^\alpha_{ljk}J^\gamma_m),$$
$$Q^{(\mu\alpha\beta\gamma)}_{imljk}=J^\mu_{im}J^\alpha_l(J^\beta_k J^\gamma_j+J^\beta_j J^\gamma_k)+2J^\alpha_{kj}J^\beta_m J^\gamma_i J^\mu_l+(J^\mu_{lk}J^\gamma_j+J^\mu_{lj}J^\gamma_k)J^\alpha_i J^\beta_m$$
$$+2(J^\alpha_{lk}J^\beta_j+J^\alpha_{lj}J^\beta_k)J^\mu_i J^\gamma_m.$$

## Reference

[1] T. Oda: On Schwarzian Derivatives in Several Variables. Kôkyûroku of R. I. M., **226**, Kyoto University (1974) (in Japanese).