145. On the Distribution of Zeros of Dirichlet's L-Function on the Line $\sigma=1/2$

By Teluhiko HILANO
Department of Mathematics, Faculty of Science and Technology, Sophia University

(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 13, 1976)

§1. **Introduction.** Results on the distribution of zeros of Dirichlet's L-function on the line $\sigma=1/2$ have been proved by analogous method in case of Riemann's ζ-function. For example, Hardy proved in 1914 that there exist infinitely many zeros of Riemann's ζ-function on the critical line and later Hardy and Littlewood proved that

$$N_0(T)>KT$$

for some absolute constant K and then these results were easily extended in case of $L(s,\chi)$. (See Süetuna [8] Chap. III.) In 1942, A. Selberg proved that

$$N_0(T)>cT\log T$$

for some constant c and this method was also applicable to $L(s,\chi)$. Recently N. Levinson gave a different proof of Selberg's result with $c=1/3$.

In this note we shall show that the essential idea of Levinson is also applicable to the case of $L(s,\chi)$ in order to prove the fundamental properties of $L(s,\chi)$. Details of the calculation will appear elsewhere.

The author wishes to express his thanks to Prof. T. Tatuzawa and Prof. Y. Motohashi who encouraged him preparing this note.

§2. **Fundamental properties of $L(s,\chi)$.** Throughout this note, χ denote a primitive character with mod q and T is a sufficiently large number. We use the following notations;

$$a=\frac{1}{2}(1-\chi(-1)) \tag{2.1}$$

$$\begin{aligned} h(s)&=h(s,\chi)\\ &=\left(\frac{\pi}{q}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right) \end{aligned} \tag{2.2}$$

$$\varepsilon(\chi)=\frac{(-i)^a}{q^{1/2}}\sum_{m=1}^{q}\chi(m)e^{2\pi i m/q} \tag{2.3}$$

$$f'(s)=h'(s)/h(s). \tag{2.4}$$

As is well known, we have

$$|\varepsilon(\chi)|=1.$$

We can choose a complex number α with

$$\bar{\alpha}=\alpha^{-1}$$

such that

$$\alpha^{-2}=\varepsilon(\chi). \tag{2.5}$$

Then we can write the functional equation of $L(s,\chi)$ as

$$\alpha h(s)L(s,\chi)=\bar{\alpha}h(1-s)L(1-s,\bar{\chi}). \tag{2.6}$$

We differentiate both sides of (2.6) and eliminate $L(1-s,\bar{\chi})$ from there. We get

$$\begin{aligned}&\alpha h(s)L(s,\chi)(f'(s)+f'(1-s))\\&\quad=-(\alpha h(s)L'(s,\chi)+\bar{\alpha}h(1-s)L'(1-s,\bar{\chi})).\end{aligned} \tag{2.7}$$

The right hand side of (2.7) is real for $s=1/2+it$. But for $|\sigma|\leqq 10$ and $t\geqq 1$, we have

$$f'(s)+f'(1-s)=\log\frac{qt}{2\pi}+O\left(\frac{1}{|t|}\right) \tag{2.8}$$

and then

$$f'(s)+f'(1-s)\neq 0$$

for sufficiently large t. Hence if, for large $t>0$, either side of (2.7) is zero, we have

$$h(s)L(s,\chi)=0.$$

But, for all s, we get

$$h(s)\neq 0,$$

then we get

$$L(s,\chi)=0.$$

Now we have just proved

Theorem A. *Let γ be a sufficiently large number. Then $\rho=1/2+i\gamma$ is a zero of $L(s,\chi)$ if and only if ρ is also a zero of* $\operatorname{Re}\alpha h(s)L'(s,\chi)$.

We can immediately prove

Corollary B. *Under the same assumption of Theorem A, if ρ is a zero of $L'(s,\chi)$, then ρ is also a zero of $L(s,\chi)$.*

Now we calculate the number of zeros $\rho=1/2+it$ of $\operatorname{Re}\alpha h(s)L'(s,\chi)$. We put

$$G(s)=G(s,\chi)=L(s,\chi)+L'(s,\chi)/(f'(s)+f'(1-s)). \tag{2.9}$$

From (2.7) and the functional equation, we get

$$\alpha h(s)L'(s,\chi)=-\bar{\alpha}h(1-s)(f'(s)+f'(1-s))G(1-s,\bar{\chi}). \tag{2.10}$$

Hence we may count the number of zeros of

$$g(t)=\operatorname{Re}\alpha h(s)(f'(s)+f'(1-s))G(s,\chi), \tag{2.11}$$

where $\sigma=1/2$, instead of $\operatorname{Re}\alpha h(s)L'(s,\chi)$. Furthermore we remark that

$$G(s,\chi)=0$$

if and only if $L'(s,\chi)=0$.

Now the zeros of $g(t)$ may occur in two cases;

i) $G(s)\neq 0$ and

$$\arg\alpha h(s)(f'(s)+f'(1-s))G(s)\equiv\frac{\pi}{2}\bmod\pi \tag{2.12}$$

ii) $G(s)=0$.

For simplicity of the following arguments, we assume that neither $t=T$ nor $T+U$ is zero of $g(t)$. There exist N_1 zeros with multiplicity and N_1' distinct zeros of $L'(s,\chi)$ on the segment $[1/2+iT, 1/2+i(T+U)]$. We divide it into $N_1'+1$ subintervals by $\rho_j'=1/2+i\gamma_j^{(1)}(\gamma_j^{(1)}<\gamma_{j+1}^{(1)})$, which are distinct zeros of $L'(s,\chi)$. Let W_j denote the change of the argument of $\alpha h(s)(f'(s)+f'(1-s))G(s)$ on the j-th subinterval. Then there exist at least

$$\sum\left(\left[\frac{W_j}{\pi}\right]-1\right)+N_1+N_1'\geqq\frac{1}{\pi}\sum W_j+N_1-N_1'-2 \tag{2.13}$$

zeros of $g(t)$ from Corollary B and above remarks. To calculate $\sum W_j$, we use the same method of Levinson. Hence we get

$$\begin{aligned}\sum W_j&=\Delta \arg h(s)+\sum V_j+O(1)\\&=\frac{U}{2}\log\frac{qT}{2\pi}+\sum V_j+O\left(\frac{U^2}{T}+1\right),\end{aligned}$$

where V_j is the change of $\arg G(s)$ along the j-th subinterval. On the other hand, we have

$$-\left(\sum V_j+\pi N_1\right)=2\pi(N_G(D)-N_1)+O(\log qT),$$

where D is the region defined by

$$\begin{gathered}1/2\leqq\sigma\leqq 3\\ T\leqq t\leqq T+U\end{gathered}$$

and $N_G(D)$ is the number of zeros of $G(s)$ in D. Hence we get

Theorem C. *We have*

$$N_0(T+U,\chi)-N_0(T,\chi)\geqq\frac{U}{2\pi}\log\frac{qT}{2\pi}-2N_G(D)+O\left(\frac{U^2}{T}+1\right).$$

§3. Main theorem. Now we may estimate $N_{\psi G}(D)$ instead of $N_G(D)$ because we need upper bound of $N_G(D)$. We put

$$\psi(s)=\sum_{n\leqq X}\frac{\chi(n)b_n}{n^s}$$

where b_n is the same as that in Levinson. Using Littlewood theorem and the approximate functional equation of $L(s,\chi)$ (See Lavrik [2], [3] or Motohashi [6]), we get similar formulas as (2.5), (2.6) and (2.20)–(2.27) of [4]. We can estimate terms corresponding to those I's as before and finally we prove

Main theorem. *For* $\varepsilon>0$, *we assume that*

$$\log q<(\log T)^{1-\varepsilon}$$

and put

$$L=\log\frac{qT}{2\pi},$$

and

$$U=\frac{T}{qL^4},\qquad X=\left(\frac{qT}{2\pi}\right)^{1/2}/(q^{5/2}L^8).$$

Then we have

$$N_0(T+U,\chi)-N_0(T,\chi)>\frac{1}{3}(N(T+U,\chi)-N(T,\chi)).$$

References

[1] P. X. Gallagher: A large sieve density estimate near $\sigma=1$. Invent. Math., **11**, 329–339 (1970).

[2] A. F. Lavrik: A functional equation for Dirichlet L-series and the problem of divisors in arithmetic progressions. A. M. S. Transl., **82**(2), 47–65 (1969).

[3] ——: An approximate functional equation for the Dirichlet L-function. Trans. Moscow Math. Soc., **18**, 101–115 (1968).

[4] N. Levinson: More than one third of zeros of Riemann zeta-function are on line $\sigma=1/2$. Advances in Math., **13**, 383–436 (1974).

[5] ——: A simplification in the proof that $N_0(T)>(1/3)N(T)$ for Riemann's zeta-function. Advances in Math., **18**, 239–242 (1975).

[6] Y. Motohashi: On an Approximate Functional Equation of Dirichlet L-function. Kokyuroku of Research Institute. Kyoto University No. 193 (1973) (in Japanese).

[7] K. Prachar: Primzahlverteilung. Springer (1957).

[8] Z. Suetuna: Analytic Theory of Numbers. Iwanami (1950) (in Japanese).

[9] Titchmarsh: The Theory of the Riemann Zeta-Function. Oxford (1951).