

## 10. A Remark on Shirota's Theorem

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(Communicated by Kenjiro SHODA, M. J. A., Jan. 12, 1977)

Let  $\Delta$  be an arbitrary nonempty set and  $R^d$  the Tihonov semifield, i.e. the ring of all real functions  $\Delta \rightarrow R$ , taken in the product topology and with the natural partial order. Next let  $X$  be a set. A function  $\rho: X^2 \rightarrow R^d$  is called a metric in  $X$  over  $R^d$  (see [3]) if it satisfies the usual axioms for a pseudometric.

The metrics over topological semifields are convenient tools for considering uniformity, proximity, and topology through a viewpoint analogous to that of classical metric spaces. For example, this viewpoint has had some reflections in a series of works written by K. Iséki and S. Kasahara [4]–[8].

The purpose of this article is to formulate the Shirota's theorem in the language of metric space, which henceforth we shall call generalized metric space (GMS), and show some of the possible generalizations.

Let  $(X, \rho)$  be a given GMS, and let  $t_\rho$  denote the natural topology generated by  $\rho$ . We say that  $\rho$  is complete or Weil-complete if every  $\xi$ -sequence is convergent (see [3]). Then there naturally arises the following

**Question A.** Is there for any metric  $\rho$  another metric  $\rho'$  such that  $t_\rho = t_{\rho'}$ , and  $\rho'$  is complete?

In the case where  $|\Delta|=1$ , i.e.  $\rho$  is a usual real metric on  $X$ , we have the following theorem, which answers Question A in the positive.

**Theorem** (Gillman and Jerison [1]). *For any real metric  $\rho: X^2 \rightarrow R$ , there exists a complete metric  $\rho': X^2 \rightarrow R^d$  such that  $t_\rho = t_{\rho'}$ .*

To give for an answer to Question A in the case where  $\Delta$  is arbitrary, we consider the following well-known process for constructing the metric  $\rho' = H(\rho)$  from the given metric  $\rho$  (see Gillman and Jerison [1]).

If  $\rho: X^2 \rightarrow R^d$  is given, then  $\rho' = H(\rho): X^2 \rightarrow R^d$  is defined by

$$\rho'(x, y) \cdot q' = |q'(x) - q'(y)|,$$

where  $\Delta' = C(X, t_\rho)$  is the family of all continuous real functions on the topological space  $(X, t_\rho)$ .

It is easy to see that  $t_\rho = t_{\rho'}$ . Now we reduce Question A to the following question about the metric  $H(\rho)$ .

**Question A'.** Under what conditions on the GMS  $(X, \rho)$ , is the

metric  $H(\rho)$  complete?

The answer is given in the following proposition:

**Proposition 1.** *If  $(X, \rho)$  is a subspace of  $R^d$  and it is complete with respect to the standard metric  $\rho(x, y) = |x - y|$ , then  $H(\rho)$  is complete.*

It is interesting to note the following.

**Proposition 1'** (see [1]). *If  $X, Y \subset R^d$  are two closed sets, then they are homeomorphic if they are isometric in the metric  $H(\rho)$  (even if  $X$  and  $Y$  are not isometric in the metric of  $R^d$ ).*

**Proposition 2** (Mackey [9]). *Let  $X$  be an arbitrary set and*

$$\rho(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y, \end{cases}$$

*then  $H(\rho)$  is complete if and only if  $|X|$  is nonmeasurable.*

**Proposition 3** (Shirota, cf. [1]). *Let  $(X, \rho)$  be a complete GMS (completely regular). Then  $H(\rho)$  is complete if and only if for every discrete subspace  $M$  of  $(X, \rho)$ , the metric  $H(\rho)$  is complete.*

**Remark.** M. Kleiber and N. Pervin [2] showed that if in the definition of GMS (cf. [3]) we omit the axiom of symmetry so that the axioms for the function  $\rho: X^2 \rightarrow R^d$  become

- 1)  $\rho(x, y) \geq 0$  for all  $x, y$  in  $X$ ,
- 2)  $\rho(x, x) = 0$  for all  $x$  in  $X$ ,
- 3)  $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$  for all  $x, y, z$  in  $X$ ,

then it is possible to metrize arbitrary topological spaces (without any axiom of separation. Cf. Antonovskii and Koshevnikova [10]).

There is little hope that an analogue of Shirota's theorem should hold in generalized metric space  $(X, \rho)$  which satisfies the axioms 1)–3) above. But adding the following two axioms to the above three, we can obtain some interesting results.

- 4)  $\rho(x, y) > 0$  if  $x \neq y$ .
- 5) For any  $x \in X$  and any  $q \in \Delta$  there exists a  $q^* \in \Delta$  such that for arbitrary  $y, z \in X$  the inequality

$$\rho(x, z) \cdot q^* + \rho(y, z) \cdot q^* \geq \rho(x, z) \cdot q$$

holds (see [11], [12] and [13]).

**Proposition 4** (cf. [11], [12] and [13]). *A topological space is regular if and only if there exists a metric  $\rho$  which satisfies the axioms 1)–5).*

The axiom 5) gives a mapping  $\varphi_\rho: \Delta \rightarrow \Delta$  defined by  $\varphi_\rho(q) = q^*$ ,  $q^* = \varphi_\rho(q, x)$ . If  $\varphi_\rho$  is the identity mapping, then  $\rho(x, y) = \rho(y, x)$  for all  $x, y$  in  $X$ . This is a characterization of completely regular spaces for which the Shirota's theorem holds.

**Question A''.** Under what conditions on the mapping  $\varphi_\rho$  the Shirota's theorem holds for regular spaces satisfying axioms 1)–5)?

Further possible generalization occurs by changing the notion of

completeness of the metric. And then the subject of investigation will be the question whether the Shirota's theorem will still hold for this generalization of the notion of completeness.

For example (see [3]), let  $\mathcal{A}$  be an arbitrary infinite set and let  $\mathcal{A}^*$  denote the set of all its finite subsets, which is ordered by inclusion. Let  $\psi: \mathcal{A}^* \setminus \mathcal{A} \rightarrow \mathcal{A}^*$  be a surjection which satisfies the following conditions:

(1)  $\psi(q^*) < q^*$  for any  $q^*$  in  $\mathcal{A}^* \setminus \mathcal{A}$ .

(2) For any  $q_1^*, q_2^*$  in  $\mathcal{A}^*$ , there exist an element  $q_3^*$  of  $\mathcal{A}^*$  and an integer  $k$  such that  $q_3^* > q_2^*$  and  $\psi^k(q_3^*) = q_1^*$ .

Now we define a new order relation  $>_\psi$  in  $\mathcal{A}^* \setminus \mathcal{A}$  as follows: The element  $q^{*'}$  is  $\psi$ -greater than the element  $q^*$  (or  $q^{*'} >_\psi q^*$ ) if there exists an integer  $k$  such that  $\psi^k(q^{*'}) = q^*$ .

A  $\mathcal{A}^*$ -sequence  $\{x_{q^*}\}$  of the elements of the space  $(X, \rho, R^d)$  is called  $\psi$ -fundamental if for any  $\alpha \in R^d_+$  there exists an element  $q_0^* \in \mathcal{A}^*$  such that  $q_1^* >_\psi q_0^*$  and  $q_2^* >_\psi q_1^*$  imply  $\rho(x_{q_1^*}, x_{q_2^*}) \succ \alpha$ .

A point  $x_0 \in X$  is called the  $\psi$ -limit for a  $\mathcal{A}^*$ -sequence  $\{x_{q^*}\}$  if for any  $\alpha \in R^d_+$  and any  $q_0^* \in \mathcal{A}^*$  there exists  $q^* >_\psi q_0^*$  such that  $\rho(x_{q^*}, x_0) \succ \alpha$ .

The space  $(X, \rho, R^d)$  is  $\psi$ -complete (in the same metric  $\rho$ ) if for any  $\psi$ -fundamental sequence  $\{x_{q^*}\}$  there exists a  $\psi$ -limit.

**Question A'''**. Under what conditions on  $\psi$  does the  $\psi$ -completeness of the metric  $\rho$  imply the completeness of  $H(\rho)$  (or its  $\psi'$ -completeness)?

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