8. On Embedding Torsion Free Modules into Free Modules*)

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Let R be a ring with identity. A right R-module M is said to be torsion free, if M is isomorphic to a submodule of a direct product of copies of $E(R_R)$, the injective hull of R_R. In [4] the author studied the following problem. What is the condition of a maximal right quotient ring Q under which every finitely generated torsion free right R-module becomes torsionless? Specializing the above problem we shall investigate rings for which every finitely generated torsion free right module is embedded into free right modules. Such a ring will be called *right T.F. ring* in this paper. In section 1 we shall give a characterization of right $T.F.$ rings in the case where Q is right self-injective.

If R is *right* QF-3 i.e., R has a unique minimal faithful right module, then, Q is right QF–3 (Tachikawa [7]), however, the converse does not hold in general. In section 2 it is proved that R is right and left QF–3, if and only if so is Q and Q is torsionless as right and left R-modules.

1. Throughout this paper R is a ring with identity and Q denotes a maximal right quotient ring of R. Let $q \in Q$. Set $(q:R)=\{r \in R; rq \in R\}$.

Proposition 1.1. *If Q is right self-injective, the following conditions are equivalent.*

(i) *Every finitely generated R-submodule of Q_R is embedded into a free right R-module.*

(ii) *Q_R is flat and $Q \otimes_R Q \cong Q$ canonically.*

Proof. (ii)⇒(i). This is obtained by the method of [4, Theorem 2].

(i)⇒(ii). Since $qR+R$ is finitely generated R-module, it is isomorphic to a submodule of $\bigoplus_{i=1}^{n} R$, finite direct sum of copies of R_R. Hence there exists $\delta_1, \delta_2, \cdots, \delta_n \in \mathrm{Hom}\,(qR+R_R, R_R)$ such that $\bigcap_{i=1}^{n} \mathrm{Ker}\,\delta_i =0$. Since δ_i is extended to $\bar{\delta}_i \in \mathrm{Hom}\,(Q_Q, Q_Q)$, $i=1,2,\cdots,n$, we can take $a_i \in Q$ so that $\delta_i(x)=a_i x$, $x \in qR+R$. Now, R has an identity.

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Therefore, $a_i \in (q:R)$. Put $M=\{a_1, \cdots, a_n\}$. Since $\bigcap_{i=1}^{n} \operatorname{Ker} \delta_i = 0$, M has no non-zero right annihilator in R and hence in Q. Put $A=\sum_{i=1}^{n} Qa_i$. Since Q_Q is injective, the finitely generated left ideal A of Q is an annihilator left ideal (cf. [2, p. 28 Theorem 8]). Then, A is a left annihilator of the right annihilator B of A. Since $B=0$, we have $A=Q$. Thus, $Q(q:R)=Q$. The consequence is immediate from Morita [5].

Let $a, b \in \bigoplus_{i=1}^{n} Q$ be arbitrary. If $a=(a_1, a_2, \cdots, a_n)$ and $b=(b_1, b_2, \cdots, b_n)$, we shall define $a \cdot b$ by $a_1b_1 + a_2b_2 + \cdots + a_nb_n \in Q$. Then, for $f \in \operatorname{Hom}\left(\bigoplus_{i=1}^{n} Q_Q, Q_Q\right)$ there exists $a \in \bigoplus_{i=1}^{n} Q$ such that $f(x)=a \cdot x$, $x \in \bigoplus_{i=1}^{n} Q$.

Theorem 1.2. *If Q is right self-injective, the following conditions are equivalent.*

(i) *R is a right T.F. ring.*

(ii) *Q_R is flat, $Q \otimes_R Q \cong Q$ canonically and Q is a right T.F. ring.*

(iii) *Q_R is flat, $Q \otimes_R Q \cong Q$ canonically and every annihilator left ideal of $M_n(Q)$, the complete ring of $n \times n$ matrix over Q, is finitely generated for every $n>0$.*

Proof. (i)$\Leftrightarrow$(ii). This is obtained by Proposition 1.1 and the proof of [4, Theorem 1].

(ii)$\Rightarrow$(iii). Let A be an annihilator left ideal of $M_n(Q)$. Set $B=\left\{(q_1, q_2, \cdots, q_n) \in \bigoplus_{i=1}^{n} Q\,;\ (q_1, q_2, \cdots, q_n) \text{ appears in a row of the matrix which belongs to } A\right\}$ and $C=\left\{c \in \bigoplus_{i=1}^{n} Q\,;\ b \cdot c = 0 \text{ for all } b \in B\right\}$. Define $\bar{b}: \bigoplus_{i=1}^{n} Q_Q \to Q_Q$ by $\bar{b}(x)=b \cdot x$, $b \in B$, $x \in \bigoplus_{i=1}^{n} Q$. Then, $\left(\bigoplus_{i=1}^{h} Q\right)/C$ is embedded into $\prod Q^{(b)}$, where $Q^{(b)}$ denotes copies of Q_Q. Since Q is right $T.F.$, there exists $f_i \in \operatorname{Hom}\left(\bigoplus_{i=1} Q_Q, Q_Q\right)$, $i=1,2,\cdots,t$, such that $\bigcap_{i=1}^{t} \operatorname{Ker} f_i = C$. Then, there exists $b_1, b_2, \cdots, b_t \in B$ such that $C=\left\{c \in \bigoplus_{i=1}^{n} Q\,;\ b_i \cdot c = 0, \text{ for all } i\right\}$. Hence the right annihilator of A in $M_n(Q)$ is the right annihilator of $\{B_1, B_2, \cdots, B_t\}$ (where B_i is the element of $M_n(Q)$ such that the first row of the matrix B_i is b_i and the other rows are zero element of $\bigoplus_{i=1}^{n} Q$) and is also a right annihilator of $\sum_{i=1}^{t} M_n(Q)B_i$. Since $M_n(Q)$ is right self-injective, $\sum_{i=1}^{t} M_n(Q)B_i$ is an annihilator left ideal. It follows that $A=\sum_{i=1}^{t} M_n(Q)B_i$.

(iii)⇒(ii). Let $\left(\bigoplus_{i=1}^{W} Q\right)/C$ be a finitely generated torsion free right Q-module, where C is a submodule of $\bigoplus_{i=1}^{n} Q$. Let $C^*=\{\beta \in M_n(Q)$; every row of the transposed matrix ${}^t\beta$ belongs to $C\}$. Since $\left(\bigoplus_{i=1}^{n} Q\right)$ C is torsionless, C^* is an annihilator right ideal of $M_n(Q)$ and hence a right annihilator of a finite subset of $M_n(Q)$. Therefore, there exists $f_i: \bigoplus_{i=1}^{n} Q_Q \to Q_Q$, $i=1, 2, \cdots, h$, such that $\bigcap_{i=1}^{h} \operatorname{Ker} f_i = C$ and hence $\left(\bigoplus_{i=1}^{n} Q\right)/C$ is contained in a free Q-module.

Remark. If R is right non-singular, Q and hence $M_n(Q)$ are regular self-injective. Then, every annihilator right ideal of $M_n(Q)$ is generated by an idempotent element and so is every annihilator left ideal. Therefore, the result of K.R. Goodearl [3, Theorem 7] is obtained from our Theorem 1. 2, too.

2. In [1] it is proved that R is right QF–3 if and only if there exists non-isomorphic simple right ideals $S_1, S_2, \cdots, S_n$ such that $\bigoplus_{i=1}^{n} E(S_i)$ is a faithful projective right ideal of R.

Proposition 2.1. *The following conditions are equivalent.*

(i) *R is right QF–3.*

(ii) *Q is right QF–3, Q is torsionless as a right R-module and $Soc\,(Q_Q)$ is an essential extension of $Soc\,(R_R)$ as a right R-module, where $Soc\,(R_R)$ is a right socle of R.*

Proof. Assume R is right QF–3. It is easily checked that $(Soc\,(R_R))Q \subset Soc\,(Q_Q)$. Let K be a simple right ideal of Q. Let eQ be the unique minimal faithful right Q-module such that $e=e_1+e_2+\cdots+e_n$, where e_i's are orthogonal primitive idempotent elements of Q and $e_iQ \cong e_jQ$ if and only if $i=j$. Since eQ is faithful, K is isomorphic to a submodule of a suitable e_iQ. Then, $e_iQ=e_iR$ implies that K contains a simple right ideal of R and hence $Soc\,(Q_Q)$ is an essential extension of Soc (R_R). Thus, (ii) holds immediately.

Conversely, assume (ii). Let $eQ=e_1Q \oplus e_2Q \oplus \cdots \oplus e_nQ$ be the same as previous. Since $Soc\,(e_iQ_Q) \cap Soc\,(R_R) \neq 0$, $Soc\,(e_iQ_R)$ is a simple right ideal of R for all i. On the other hand, since e_iQ is torsionless R-module, e_iQ is isomorphic to a right ideal I_i of R which contains a simple right ideal. Now, $i \neq j$ implies I_i and I_j are non-isomorphic, then $\sum_{k=1}^{n} I_k = \bigoplus_{k=1}^{n} I_k$ in R and R is right QF–3.

In the following right and left QF–3 rings are called QF–3.

Theorem 2.2. *The following conditions are equivalent.*

(i) *R is QF–3.*

(ii) *Q is QF–3 and Q is torsionless as right and left R-modules*

(iii) *Q is a QF–3 maximal two-sided quotient ring of R and R has a minimal dense right ideal and a minimal dense left ideal.*

Proof. From [4, Proposition 2] it is not hard to see that the above conditions (i) or (ii) implies Q is also a maximal left quotient ring. (i)$\Rightarrow$(iii) is obtained by E.A. Rutter Jr. [6, Corollary 1.2 and Theorem 1.4].

(iii)$\Rightarrow$(ii). Let J be a minimal dense left ideal of R. Since $\bigcap_{q\in Q}(q:R)$ is a dense left ideal, it contains J and $JQ\subset R$. Let $p\in Q$ be a non-zero element. There exists $j\in J$ such that $jp\neq 0$. Define $\vec{j}\in \mathrm{Hom}\,(Q_R, R_R)$ by left multiplication of j. Since $\vec{j}(p)\neq 0$, Q_R is torsionless.

(ii)$\Rightarrow$(i). By Proposition 2.1 it is sufficient to prove that $Soc\,(Q_Q)_R$ is an essential extension of $Soc\,(R_R)$. Let K be a simple right ideal of Q. Since ${}_RQ$ is torsionless, there exists $\delta\in \mathrm{Hom}\,({}_RQ, {}_RR)$ such that $\mathrm{Ker}\,\delta\not\supset K$. Clearly δ is a right multiplication of an element of $\bigcap_{q\in Q}(R:q)$, where $(R:q)=\{r\in R\,;\,qr\in R\}$. Set $M=K\left(\bigcap_{q\in Q}(R:q)\right)$. Then, the above observation implies $M\neq 0$. Let $N\neq 0$ be an R-submodule of M. Since K is a simple right ideal of Q, $NQ=K$. Hence $M=NQ\left(\bigcap_{q\in Q}(R:q)\right)\subset N$ and it follows that M is a simple right ideal of R and the consequence is immediate.

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