6. *A Note on the Large Sieve*

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1. Let $a(n)$ be arbitrary complex numbers. Let $c_r(n)$ and $\varphi(n)$ be the Ramanujan sum and the Euler function, respectively. Then a slight modification of a recent large sieve inequality of Selberg [1; Théorème 7A] states that we have, uniformly for any Q, R, M, N, k, l,

$$(\sharp)\qquad \sum_{\substack{q\leqq Q,\ r\leqq R\\ (q,r)=(qr,k)=1}} \frac{q}{\varphi(qr)} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{M\leqq n<M+N\\ n\equiv l\ (\mathrm{mod}\ k)}} \chi(n)c_r(n)a(n)\Bigg|^2 \leqq (N/k+(QR)^2) \sum_{\substack{M\leqq n<M+N\\ n\equiv l\ (\mathrm{mod}\ k)}} |a(n)|^2,$$

where $\sum^*$ denotes as usual a sum over primitive characters $\chi\ (\mathrm{mod}\ q)$. This, in case of $k=1$, has an important application to Dirichlet's L-functions (see [1; p. 40 and p. 83] and also [6] [3]), but in the present note we are concerned with its sieve-effect. As is easily seen, ($\sharp$) implies the linear sieve result of Bombieri-Davenport [1; Théorème 8] and thus the Brun-Titchmarsh theorem. On the other hand the B-T theorem has recently got some improvements (see [4] and also [5] [2] [7]). So, noticing the fact that the dual of ($\sharp$), in case of $Q=1$, is by virtue of $c_r(n)$ reduced to the form similar to the classical sieve idea of Selberg, we may well expect that ($\sharp$) can be improved so as to contain our improvements of the T-B theorem. Then we shall have a first example of large sieve inequalities sensitive to arithmetic progressions.

Now we announce such an improvement of ($\sharp$):

Theorem. *If* $(k,l)=1$, *then we have*

$$\sum_{\substack{q\leqq Q,\ r\leqq R\\ (q,r)=(qr,k)=1}} \frac{q}{\varphi(qr)} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{n\leqq N\\ n\equiv l\ (\mathrm{mod}\ k)}} \chi(n)c_r(n)a(n)\Bigg|^2 \leqq \Lambda \sum_{\substack{n\leqq N\\ n\equiv l\ (\mathrm{mod}\ k)}} |a(n)|^2,$$

where, ε being an arbitrary small positive constant,

$$\Lambda=\frac{N}{k}(1+O((\log N)^{-1}))+O_\varepsilon\left(\frac{QR^{1+\varepsilon}}{\sqrt{k}}(R+kQ^2)(\log N)^4\right).$$

Corollary. *Let p denote a prime and let $\pi(N;k,l)$ be the number of primes$\equiv l\ (\mathrm{mod}\ k)$ less than N. Then we have, under the condition* $N^{2/5}\geqq Q^2k$,

$$\sum_{\substack{q\leqq Q\\ (q,k)=1}} \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{p\equiv l\ (\mathrm{mod}\ k)\\ p\leqq N}} \chi(p)\Bigg|^2 \leqq (2+\varepsilon)\frac{N}{\varphi(k)\log(N/(\sqrt{k}\,Q))}\pi(N;k,l).$$

Further improvements of (♯), which implies the result of [2] as a special case, can be obtained by using Burgess' estimate of character sums. The detailed account will be published elsewhere.

2. Here we indicate very briefly the outline of the proof of our theorem. First we note that it is sufficient to prove the dual inequality:

$$B(N)=\sum_{\substack{n\leqq N\\ n\equiv l \pmod k}}\left|\sum_{\substack{q\leqq Q,\, r\leqq R\\ (q,r)=(qr,k)=1}}\left(\frac{q}{\varphi(qr)}\right)^{1/2}\sideset{}{^*}\sum_{\chi \pmod q}\chi(n)c_r(n)b(\chi,r)\right|^2$$
$$\leqq \Lambda \sum_{\substack{q\leqq Q,\, r\leqq R\\ (q,r)=(qr,k)=1}}\sideset{}{^*}\sum_{\chi \pmod q}|b(\chi,r)|^2,$$

where $b(\chi,r)$ are arbitrary complex numbers. And then it is better to consider, instead, the first Riesz mean $B_1(N)$ of $B(N)$, since $B_1(N)$ admits the analytic expression:

$$B_1(N)=\frac{1}{2\pi i\varphi(k)}\sum_{\xi \pmod k}\bar{\xi}(l)\int_{(2)}\Phi(s,\xi)\frac{N^s}{s^2}ds,$$

where $\Phi(s,\xi)$ with a Dirichlet character ξ is given explicitly in Lemma 1 below. Then we appeal to the following lemmas and by the smoothening device we reach the above dual result.

Lemma 1. *Let $\mu(u)$ be the Möbius function and let*

$$\lambda(\chi,d)=d\sum_{\substack{u\leqq R/d\\ (u,qk)=1}}\mu(u)(\varphi(du))^{-1/2}b(\chi,du),$$

where $\chi \pmod q$. Further let

$$G_s(\chi,d)=\bar{\chi}(d)d^s\prod_{p|d}(1-\chi(p)p^{-s})$$

and put

$$H(s;\chi_1,\chi_2;\xi)=\sum_{d\leqq R}d^{-2s}G_s(\chi_1\bar{\chi}_2\xi,d)\chi_1\bar{\chi}_2\xi(d^2)$$
$$\times\left(\sum_{u\leqq R/d}u^{-s}\lambda(\chi_1,du)\chi_1\bar{\chi}_2\xi(u)\right)\left(\sum_{v\leqq R/d}v^{-s}\overline{\lambda(\chi_2,dv)}\chi_1\bar{\chi}_2\xi(v)\right).$$

Then we have

$$\Phi(s,\xi)=\sum_{\substack{q_1,q_2\leqq Q\\ (q_1q_2,k)=1}}\left(\frac{q_1q_2}{\varphi(q_1)\varphi(q_2)}\right)^{1/2}\sideset{}{^*}\sum_{\chi_1 \pmod{q_1}}\sideset{}{^*}\sum_{\chi_2 \pmod{q_2}}L(s,\chi_1\bar{\chi}_2\xi)H(s;\chi_1,\chi_2;\xi),$$

where $L(s,\chi)$ is Dirichlet's L-function.

Lemma 2. *$\Phi(s,\xi)$ is entire, except for $\xi=\xi_0$ principal. And the residue at $s=1$ of $\Phi(s,\xi_0)$ is*

$$\frac{\varphi(k)}{k}\sum_{\substack{q\leqq Q,\, r\leqq R\\ (r,q)=(rq,k)=1}}\sideset{}{^*}\sum_{\chi \pmod q}|b(\chi,r)|^2.$$

Lemma 3. *We have*

$$\sum_{\substack{q\leqq Q\\ (q,k)=1}}\sideset{}{^*}\sum_{\chi \pmod q}\sum_{\xi \pmod k}\left|\sum_{M\leqq n<M+N}\chi\xi(n)a(n)\right|^2$$
$$\leqq(N+kQ^2)\sum_{\substack{M\leqq n<M+N\\ (n,k)=1}}|a(n)|^2.$$

References

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