2. *A Remark on Fractional Powers of Linear Operators in Banach Spaces*

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1. Introduction. Let X be a Banach space and A be a densely defined, closed linear operator in X satisfying

(1) the resolvent set $\rho(-A)$ of $-A$ contains the non-negative real axis and

(2) $$\|\lambda(\lambda+A)^{-1}\|\leqq M \qquad \text{for } \lambda>0,$$

or equivalently,

$(2)_1$ $$\|\lambda(\lambda+A)^{-1}\|\leqq M_1 \qquad \text{for } |arg\lambda|\leqq\omega$$

holds, where M, M_1 and ω are some positive constants independent of λ. As is well known, the fractional power A^α, $0<\alpha<1$ of A is defined through

(3) $$A^{-\alpha}=\frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda-A)^{-1}d\lambda,$$

where Γ runs in $\rho(A)$ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ $(\pi-\omega\leqq\theta\leqq\pi)$ avoiding the non-positive real axis.

The purpose of the present paper is to describe a criterion for the width of the domain $D(A^\alpha)$ of A^α, and then apply it to an evolution equation of parabolic type:

$$du(t)/dt+A(t)u(t)=f(t), \qquad 0\leqq t\leqq T.$$

2. Basic theorem. We denote by $D(A_\alpha)$, $0<\alpha<1$ the set of all $x\in X$ such that $\int_\Gamma \lambda^{\alpha-1}A(\lambda-A)^{-1}xd\lambda$ is absolutely convergent and define a linear operator A_α by

(4) $$A_\alpha x=\frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1}A(\lambda-A)^{-1}xd\lambda, \qquad x\in D(A_\alpha).$$

In view of (3) it is evident that $D(A)$ is contained in $D(A_\alpha)$, $0<\alpha<1$.

Lemma. *If $x\in X$ and*

$(5)_1$ *$\lambda^\beta A(\lambda+A)^{-1}x$, $|arg\lambda|\leqq\omega$ is uniformly bounded for some $0<\beta\leqq1$, then $x\in D(A^\alpha)$ and $A^\alpha x=A_\alpha x$ for any α with $0<\alpha<\beta$.*

Proof. Clearly $x\in D(A_\alpha)$, $0<\alpha<\beta$ and (4) holds good. From

$$A^{-1}A^{-\alpha}A_\alpha x=A^{-\alpha}A^{-1}A_\alpha x=A^{-\alpha}\frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1}(\lambda-A)^{-1}xd\lambda$$
$$=A^{-\alpha}A^{\alpha-1}x=A^{-1}x,$$

it follows that $A^{-\alpha}A_\alpha x=x$, which implies that $x\in D(A^\alpha)$ and $A^\alpha x=A_\alpha x$.

Theorem. *Let A be a densely defined, closed linear operator satis-*

fying (1) *and* (2). *In order that an* $x \in X$ *belong to* $D(A^\alpha)$ *and*

$$A^\alpha x = \frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1}A(\lambda-A)^{-1}x\,d\lambda$$

hold for some $0<\alpha<1$, *it is necessary and sufficient that* (5) $\lambda^\beta A(\lambda+A)^{-1}x$, $\lambda>0$ *is uniformly bounded for some* β *with* $0<\beta\leqq 1$.

Proof. *The necessity.* A simple calculation shows that $\|\lambda^\alpha A^{-\alpha}A(\lambda+A)^{-1}\|$, $\lambda>0$ is bounded. Hence if $x\in D(A^\alpha)$, $0<\alpha<1$, then (5) holds with $\beta=\alpha$.

The sufficiency. If an $x\in X$ satisfies (5) with $0<\beta\leqq 1$, it also satisfies $(5)_1$ with some small $\omega>0$. Therefore by Lemma we conclude that $x\in D(A^\alpha)$ and $A^\alpha x = A_\alpha x$ for $0<\alpha<\beta$. Q.E.D.

Remark. In the above we have proved that

$$D(A_\beta)\subset D(A^\beta)\subset B_\beta\subset D(A_\alpha)\subset D(A^\alpha)\subset B_\alpha$$

as long as $0<\alpha<\beta<1$, where B_β, $0<\beta\leqq 1$ is the set of $x\in X$ which satisfy (5) or equivalently $(5)_1$ for some small $\omega>0$.

3. Application. In this section we consider a system $A(t)$, $t\in[0,T]$ of densely defined closed linear operators satisfying (1) and (2) (or $(2)_1$) with M (M_1 and ω) independent of t. We assume that $A(t)^{-1}$ is strongly continuously differentiable in $t\in[0,T]$.

Proposition. *Under the above assumptions the followings are equivalent:*

(6) *the range* $R\left(\frac{d}{dt}A(t)^{-1}\right)$ *of* $\frac{d}{dt}A(t)^{-1}$ *is included with* $D(A(t)^\rho)$ *and*

$$\left\|A(t)^\rho\frac{d}{dt}A(t)^{-1}\right\|,\ t\in[0,T] \text{ is bounded for some } 0<\rho<1;$$

(7) $\left\|\lambda^{\rho_1}A(t)(\lambda+A(t))^{-1}\frac{d}{dt}A(t)^{-1}\right\|$, $t\in[0,T]$, $|arg\lambda|\leqq\omega$ *is bounded for some* $0<\rho_1\leqq 1$.

Remark. H. Tanabe [2] constructed the solution to the evolution equation

(8) $$du(t)/dt + A(t)u(t) = f(t), \qquad 0\leqq t\leqq T$$

by assuming $\omega>\pi/2$ (parabolicity) and (6). Recently A. Yagi [3] constructed the solution of (8) under the assumption (7) instead of (6). But his method, which makes no explicit use of the fractional power of $A(t)$, is new and seriously interesting for the author.

Proof of Proposition. The implication (6)→(7) is clear. For the proof of (7)→(6) we apply the lemma stated above to know that $\frac{d}{dt}A(t)^{-1}x$, $x\in X$ belongs to $D(A(t)^\rho)$ for any $0<\rho<\rho_1$ and that the integral

$$A(t)^\rho\frac{d}{dt}A(t)^{-1}x = \frac{1}{2\pi i}\int_\Gamma \lambda^{\rho-1}A(t)(\lambda-A(t))^{-1}x\,d\lambda$$

is absolutely convergent uniformly in $t\in[0,T]$.

4. **Example.** Finally we deal with an example of $A(t)$, $t\in[0,T]$. Let $A(t)$ be for each $t\in[0,T]$ the associated operator with a *regular elliptic* boundary value problem $(A(t,x;D),\{B_j(t,x;D)\}_{j=1}^m,G)$ of order $2m$ in R^n, where

$$G\subset R^n,\qquad D=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n},$$
$$A(t,x;D)=\sum_{|\alpha|\leqq 2m}a_\alpha(t,x)D^\alpha,\qquad B_j(t,x;D)=\sum_{|\beta|\leqq m_j}b_{j\beta}(t,x)D^\beta,$$
$$0\leqq m_j\leqq 2m-1,\qquad j=1,\cdots,m.$$

For the details we refer the reader to S. Agmon [1]. $A(t)$ is defined as usual by $(A(t)u)(x)=A(t,x;D)u(x)$ for any $u\in D(A(t))$, the set of all $u\in W_p^{2m}(G)$ such that $B_j(t,x;D)u(x)=0$, $x\in\partial G$, $j=1,\cdots,m$. We suppose on $(A(t,x;D),\{B_j(t,x;D)\}_{j=1}^m,G)$ Agmon's condition for $\theta\notin(-\theta_0,\theta_0)$, which furnishes $A(t)$ with the following property: $\rho(-A(t))$ includes a sector $|arg\lambda|\leqq\pi-\theta_0$, $|\lambda|\geqq C$, where

$$\|u\|_{2m,p}+|\lambda|\,\|u\|_p$$
$$\leqq C\Big\{\Big\|\Big(\lambda+A(t,\cdot\,;D)\Big)u\Big\|_p+\sum_{j=1}^m|\lambda|^{1-m_j/2m}\|g_j\|_p+\sum_{j=1}^m\|g_j\|_{2m-m_j,p}\Big\},\tag{9}$$
$$g_j(x)=B_j(t,x;D)u(x),\qquad x\in\partial G,\ j=1,\cdots,m$$

holds for any $u\in W_p^{2m}(G)$.

We will prove here with the aid of the above proposition that $A(t)$, $t\in[0,T]$ *satisfies* (6) *if*

(10) $a_\alpha(t,x)$, $|\alpha|\leqq 2m$; $D^\gamma b_{j\beta}(t,x)$, $|\beta|\leqq m_j$, $|\gamma|\leqq 2m-m_j$, $j=1,\cdots,m$ *are continuously differentiable in* t *in* $[0,T]\times\overline{G}$.

It is not difficult to verify that the strong continuous differentiability of $A(t)^{-1}$ in $t\in[0,T]$ follows from (10). We replace $A(t)+kI$ by $A(t)$ for a suitable number k if necessary. Putting

$$u(t)=A(t)^{-1}g,\qquad v_\lambda(t)=\lambda(\lambda+A(t))^{-1}du(t)/dt$$

for any $g\in L_p(G)$, we have

$$(\lambda+A(t,x;D))\Big(\frac{\partial}{\partial t}u(t,x)-v_\lambda(t,x)\Big)=-\sum_{|\alpha|\leqq 2m}\Big(\frac{\partial}{\partial t}a_\alpha(t,x)\Big)D^\alpha u(t,x),\qquad x\in G,$$

$$B_j(t,x;D)\Big(\frac{\partial}{\partial t}u(t,x)-v_\lambda(t,x)\Big)=\sum_{|\beta|\leqq m_j}\Big(\frac{\partial}{\partial t}b_{j\beta}(t,x)\Big)D^\beta u(t,x),\qquad x\in\partial G,\ j=1,\cdots,m.$$

Making use of (9) we have

$$\Big\|\lambda A(t)(\lambda+A(t))^{-1}\frac{d}{dt}A(t)^{-1}g\Big\|_p=|\lambda|\,\Big\|\frac{d}{dt}u(t)-v_\lambda(t)\Big\|_p$$
$$\leqq C\Big(1+\sum_{j=1}^m|\lambda|^{1-m_j/2m}\Big)\|u(t)\|_{2m,p}\leqq C\Big(1+\sum_{j=1}^m|\lambda|^{1-m_j/2m}\Big)\|g\|_p$$

for any $t\in[0,T]$ and λ with $|arg\lambda|\leqq\pi-\theta_0$, $|\lambda|\geqq C$, where, as in (9), we use C to denote a positive constant independent of t and λ. But $B_j(t,x;D)$ is, if $m_j=0$, considered to be the identity operator and hence we obtain

$$\left\| A(t)(\lambda + A(t))^{-1}\frac{d}{dt}A(t)^{-1} \right\|_{L_p(G)\to L_p(G)} \tag{11}$$
$$\leqq C\left(|\lambda|^{-1} + \sum_{m_j \neq 0} |\lambda|^{-m_j/2m}\right).$$

Clearly (11) implies (7) with $\omega=\pi-\theta_0$ and

$$\rho_1 = \begin{cases} \operatorname{Min}\limits_{m_j\neq 0} (m_j/2m) & ((m_1, \cdots, m_m)\neq(0, \cdots, 0)), \\ 1 & ((m_1, \cdots, m_m)=(0, \cdots, 0)). \end{cases}$$

Thanks to the above proposition, we have proved that (6) is valid for our $A(t)$, $t\in[0, T]$ with any positive $\rho<\rho_1$; $\rho=1/2mp$, $p>1$ for example.

References

[1] S. Agmon: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math., **15**, 119–147 (1962).

[2] H. Tanabe: Note on singular perturbation for abstract differential equations. Osaka J. Math., **1**, 239–252 (1964).

[3] A. Yagi: On the abstract linear evolution equations in Banach spaces. J. Math. Soc. Japan, **28**, 290–303 (1976).