37. The Singular Cauchy Problem for Systems whose Characteristic Roots are Non Uniform Multiple

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1977)

1. Introduction. In [3], applying the method of B. Granoff-D. Ludwig [1], the author studied the singularities of the solutions of the Cauchy problem for a certain system which has a pair of characteristic roots with non uniform multiplicities. It was Duhamel's principle which played a fundamental role in [1] and [3]. The aim of this note is to show that a thorough use of Duhamel's principle enables us to generalize the result of [3] in case more than two characteristic roots are non uniform multiple. Since the calculations become more complicated as the number of the characteristic roots with non uniform multiplicities increases, we only treat the simplest case which three characteristic roots are non uniform multiple. Therefore we must prepare another proof when arbitrary number of characteristic roots are non uniform multiple. However, one may easily conjecture the theorem concerning the general case from this note.

2. Assumptions and result. Let $(t,x)=(t,x_1,\cdots,x_n)\in \boldsymbol{C}^{n+1}$ and $\xi=(\xi_1,\cdots,\xi_n)$ be the covector at $x=(x_1,\cdots,x_n)$.

Consider the system:

$$\mathcal{L}u=\partial u/\partial t+\sum_{\mu=1}^{n} A^{\mu}(t,x)\partial u/\partial x_{\mu}+B(t,x)u=0, \tag{2.1}$$

where A^{μ}, B are $k\times k$-matrices holomorphic in a neighborhood of the origin. We impose a Cauchy data which has a pole on $x_1=0$ and denote this Cauchy problem by (CP).

Now we assume the following assumptions (I) ~ (V).

(I) For any $(t,x)\sim 0$ and $\xi\sim(1,0,\cdots,0)$, the matrix $-\sum_{\mu=1}^{n} A^{\mu}(t,x)\xi_{\mu}$ has k (counting multiplicities) eigenvalues $\lambda^l(t,x;\xi)$ $(1\le l\le k)$ and the associated eigenvectors $R^l(t,x;\xi)$ $(1\le l\le k)$ form a complete set.

(II) For any $(t,x)\sim 0$ and $\xi\sim(1,0,\cdots,0)$, each set of eigenvalues $\{\lambda^1,\lambda^l; 4\le l\le k\}$, $\{\lambda^2,\lambda^l; 4\le l\le k\}$, $\{\lambda^3,\lambda^l; 4\le l\le k\}$ is mutually distinct.

(III) λ^l, R^l $(1\le l\le 3)$ are holomorphic in $(t,x)\sim 0$, $\xi\sim(1,0,\cdots,0)$.

(IV) The Poisson brackets $\{\lambda^i,\lambda^j\}$ $(1\le i<j\le 3)$ vanish for $(t,x)\sim 0$, $\xi\sim(1,0,\cdots,0)$.

In order to state the assumption (V), we define the phases $F^i(t,x)$ $(1\le i\le k)$ and the multi-phases $F^{ij}(t,x;s_1)$ $(1\le i<j\le 3)$, $F^{123}(t,x;s_1,s_2)$ by

$$\begin{cases} F^i_t=\lambda^i(t,x;\nabla_x F^i),\ F^i(0,x)=x_1\ (1\le i\le k),\\ F^{ij}_t=\lambda^j(t,x;\nabla_x F^{ij}),\ F^{ij}(s_1,x,s_1)=F^i(s_1,x)\ (1\le i<j\le 3),\\ F^{123}_t=\lambda^3(t,x;\nabla_x F^{123}),\ F^{123}(s_2,x;s_1,s_2)=F^{12}(s_2,x;s_1).\end{cases}$$

Then the assumption (V) is as follows.

(V) $F^{ij}(0,0,s_1)$ $(1\le i<j\le 3)$, $F^{123}(0,0,0,s_2)\not\equiv 0$. Additionally, let $\omega(t,x,s_1)$ be the discriminant of $F^{123}(t,x,s_1,s_2)$, then $\omega(0,0,s_1)\not\equiv 0$.

Now we define K_i $(1\le i\le k)$, K_{ij} $(1\le i<j\le 3)$, K_{123} by

K_i; $F^i(t,x)=0$ $(1\le i\le k)$, K_{ij}; $\omega^{ij}(t,x)=0$ $(1\le i<j\le 3)$,

K_{123}; $\Delta(t,x)=0$,

where $\omega^{ij}(t,x)$ $(1\le i<j\le 3)$, $\Delta(t,x)$ are the discriminants of $F^{ij}(t,x,s_1)$ $(1\le i<j\le 3)$, $\omega(t,x,s_1)$. Then set $K=\left(\bigcup_{i=1}^{k}K_i\right)\cup\left(\bigcup_{1\le i<j\le 3}K_{ij}\right)\cup K_{123}$.

In these situations we have the following theorem.

Theorem. *Under the assumptions* (I)~(V), *there exists a neighborhood V of the origin such that the Cauchy problem* (CP) *has a unique holomorphic solution u(t, x) on the simply connected covering space over V−K. Moreover u(t, x) is given by*

$$\begin{aligned} u(t,x)=&\sum_{i=1}^{k}\{G^i(t,x)/[F^i(t,x)]^{q_i}+H^i(t,x)\log F^i(t,x)\}\\ &+\sum_{1\le i<j\le 3}\int_0^t\{G^{ij}(t,x;s_1)/[F^{ij}(t,x;s_1)]^{q_{ij}}\\ &\qquad+H^{ij}(t,x;s_1)\log F^{ij}(t,x;s_1)\}ds_1\\ &+\int_0^t\int_0^{s_2}\{G^{123}(t,x;s_1,s_2)/[F^{123}(t,x;s_1,s_2)]^{q_{123}}\\ &\qquad+H^{123}(t,x;s_1,s_2)\log F^{123}(t,x;s_1,s_2)\}ds_1ds_2\\ &+H(t,x),\end{aligned}\tag{2.2}$$

where G^i, H^i $(1\le i\le k)$, G^{ij}, H^{ij} $(1\le i<j\le 3)$, G^{123}, H^{123}, H *are vector functions holomorphic at the origin and* q_i, q_{ij}, q_{123} *are positive integers.*

Remark. The assumption (V) is unnecessary to obtain the expression (2.2). Under the assumption (V), it is easy to prove that the singularities of the solution $u(t,x)$ lie on K.

3. Outline of the proof. We only illustrate the construction of an asymptotic (a formal) solution, because its convergence (thus its exactness) can be proved by the method of majorant due to Y. Hamada-J. Leray-C. Wagschal [2] and others. Moreover, for simplicity, we only construct an asymptotic solution corresponding to the characteristic roots λ^l $(1\le l\le 3)$.

We seek the formal solution in the following form:

$$\begin{aligned} u(t,x)=&\sum_{l=1}^{3}\sum_{j=0}^{\infty}f_j(F^l(t,x))a^l_j(t,x)\\ &+\sum_{1\le l<m\le 3}\sum_{j=0}^{\infty}\int_0^t f_j(F^{lm}(t,x;s_1))b^{lm}_j(t,x;s_1)ds_1\\ &+\sum_{j=0}^{\infty}\int_0^t\int_0^{s_2}f_j(F^{123}(t,x;s_1,s_2))c_j(t,x;s_1,s_2)ds_1ds_2,\end{aligned}\tag{3.1}$$

where $\{f_j(\zeta)\}_{j=-1}^{\infty}$ are the wave forms which satisfy $d/d\zeta\, f_j(\zeta)=f_{j-1}(\zeta)$ $(j\geq 0)$.

Set

$$b_j^{lm}(t,x;s_1)=\beta_{jl}^{lm}(t,x;s_1)R^l(t,x;\nabla_x F^{lm}(t,x;s_1))+\beta_{jm}^{lm}R^m,$$

$$c_j(t,x;s_1,s_2)=\sum_{k=1}^{3}\gamma_j^k(t,x;s_1,s_2)R^k(t,x;\nabla_x F^{123}(t,x;s_1,s_2)).$$

Substitute (3.1) into (2.1) and seek the conditions in order that (3.1) will be a formal solution. Then we obtain the following series of recurrence relations. Namely, with the conventions $\alpha_j^l=0$, $\beta_{jk}^{lm}=0$, $\gamma_j^k=0$ for $j<0$ and $\beta_{jk}^{lm}=\beta_{jk}^{ml}$, $A_k=F_t^k+\sum_{\mu=1}^{n}A^\mu F_{x_\mu}^k$ $(1\leq k\leq 3)$, we have

$$(3.2)\quad \begin{cases} A_k\alpha_{j+1}^k+\mathcal{L}(\alpha_j^k)+\beta_{jl}^{kl}(t,x;y_{kl})R^l+\beta_{jm}^{km}(t,x;y_{km})R^m \\ \quad +\sum' M_x^1[\beta_{j-1l}^{km}](t,x;y_{klm})=0 \qquad \text{for } \{k,l,m\}=\{1,2,3\},\ l<m, \end{cases}$$

$$\begin{cases} (\partial/\partial t+\partial/\partial s_1)\beta_{jk}^{kl}+M_x^1[\beta_{jk}^{kl}]+M^0[\beta_{jl}^{kl}]+M_x^1[\beta_{j-1k}^{kl}]+M_{x,s_1}^2[\beta_{j-1l}^{kl}] \\ \qquad\qquad +M_x^1[\gamma_{j-1}^m](z_k)=0, \\ \partial/\partial t\ \beta_{jl}^{kl}+M_x^1[\beta_{jl}^{kl}]+M^0[\beta_{jm}^{kl}]+M_{x,s_1}^2[\beta_{j-1l}^{kl}]+M_x^1[\beta_{j-1m}^{kl}]+M_x^1[\gamma_{j-1}^m](z_k) \\ \qquad\qquad =0, \\ \gamma_j^m(z_k)=\sum_{d=k,l}M_x^1[\beta_{jd}^{kl}]+M_x^1[\beta_{j-1d}^{kl}]+\sum_{d=k,l}M_x^1[\gamma_{j-1}^d](z_k) \\ \qquad\qquad \text{for } \{k,l,m\}=\{1,2,3\},\ k<l, \end{cases}$$

$$(3.3)\quad \begin{cases} (\partial/\partial t+\partial/\partial s_1+\partial/\partial s_2)\gamma_j^1+M_x^1[\gamma_j^1]+\sum_{k=2,3}M^0[\gamma_j^k]+M_x^1[\gamma_{j-1}^1]+M_{x,s_1}^2[\gamma_{j-1}^2] \\ \qquad\qquad +M_{x,s_1,s_2}^2[\gamma_{j-1}^3]=0, \\ (\partial/\partial t+\partial/\partial s_2)\gamma_j^2+M_x^1[\gamma_j^2]+\sum_{k=1,3}M^0[\gamma_j^k]+M_{x,s_1}^2[\gamma_{j-1}^1]+M_x^1[\gamma_{j-1}^2] \\ \qquad\qquad +M_{x,s_2}^2[\gamma_{j-1}^3]=0, \\ \partial/\partial t\ \gamma_j^3+M_x^1[\gamma_j^3]+\sum_{k=1,2}M^0[\gamma_j^k]+M_{x,s_1,s_2}^2[\gamma_{j-1}^1]+M_{x,s_2}^2[\gamma_{j-1}^2] \\ \qquad\qquad +M_x^1[\gamma_{j-1}^3]=0, \end{cases}$$

where the summation $\sum'$ is taken over $k,l,m\in\{1,2,3\}$ such that $k=l$, $m\neq l$ or $m=l$, $k\neq l$ and $y_{12}=y_{13}=y_{23}=t$, $y_{21}=y_{31}=y_{32}=0$, $y_{221}=y_{223}=y_{311}=y_{322}=y_{331}=y_{332}=0$, $y_{112}=y_{113}=y_{122}=y_{133}=y_{211}=y_{233}=t$, $z_1=(t,x;s_1,t)$, $z_2=(t,x;s_1,s_1)$, $z_3=(t,x;0,s_1)$. Moreover, M_Δ^p stands for holomorphic linear partial differential operator of total order p with respect to the variable Δ and its order with respect to the variables s_1,s_2 are not greater than one.

In deriving the above recurrence relations, we have used the following formulas:

$$\begin{cases} F_{s_1}^{ij}(t,x;s_1)=\lambda^i(t,x;\nabla_x F^{ij}(t,x;s_1))-\lambda^j(t,x;\nabla_x F^{ij}) \qquad (1\leq i<j\leq 3) \\ F_{s_1}^{123}(t,x;s_1,s_2)=\lambda^1(t,x;\nabla_x F^{123}(t,x;s_1,s_2))-\lambda^2(t,x;\nabla_x F^{123}), \\ F_{s_2}^{123}(t,x;s_1,s_2)=\lambda^2(t,x;\nabla_x F^{123}(t,x;s_1,s_2))-\lambda^3(t,x;\nabla_x F^{123}), \\ F^{ij}(t,x;t)=F^i(t,x),\ F^{ij}(t,x;0)=F^j(t,x) \qquad (1\leq i<j\leq 3), \\ F^{123}(t,x;s_1,t)=F^{12}(t,x;s_1),\ F^{123}(t,x;0,s_2)=F^{23}(t,x;s_2), \\ F^{123}(t,x;s_2,s_2)=F^{13}(t,x;s_2),\ L^iA^\mu R^j=-(\lambda^j-\lambda^i)L^i\partial R^j/\partial\xi_\mu, \end{cases}$$

where $\{L^l;1\leq l\leq k\}$ is the dual basis of $\{R^l;1\leq l\leq k\}$.

Now, as usual, we can see that α_j^k $(1 \leq k \leq 3)$ are determined successively from (3.2). As for β_{jm}^{kl} $(1 \leq k < l \leq 3,\ m = k, l)$, the location of the initial manifolds are all non-characteristic to the differential equations defining β_{jm}^{kl} $(1 \leq k < l \leq 3,\ m = k, l)$ if we seek the initial conditions for β_{jm}^{kl} $(1 \leq k < l \leq 3,\ m = k, l)$ from (3.2). Similarly, we also see that γ_j^k $(1 \leq k \leq 3)$ are determined successively from (3.3). Thus we have constructed an asymptotic solution.

References

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