§ 1. Introduction. A non-singular algebraic surface $S$ is called an Enriques surface if the following two conditions are satisfied:

(i) The geometric genus and the irregularity both vanish.

(ii) If $K$ is a canonical divisor on $S$, $2K$ is linearly equivalent to 0.

Historically speaking Enriques surfaces were the first example of non-rational algebraic surfaces which satisfy the above condition (i). In this paper we are mainly interested in Enriques surfaces over the field of complex numbers $C$.

From the condition (ii), it follows that there exists a two-sheeted unramified covering $\pi: T \to S$ such that $T$ is a $K3$ surface. Since every $K3$ surface is known to be simply-connected by Kodaira [6], $T$ is the universal covering of $S$. We take a holomorphic 2-form $\psi$ on $T$ which is non-zero everywhere, and consider the integrals

$$\int \psi \quad \text{for } \gamma \in H_2(T, \mathbb{Z}).$$

We let $\tau$ denote the covering transformation $T \to T$ over $S$ so that $\tau^2 = \text{id}$. Since $S$ has no holomorphic 2-form, we have $\tau^* \psi = -\psi$. On the other hand, $\tau$ acts on $H_2(T, \mathbb{Z})$ as an involution. If $\gamma$ is invariant by $\tau$, then the above integral (1) vanishes. Therefore the periods of $\psi$ are determined by the integrals (1) over those 2-cycles $\gamma$ satisfying $\tau \gamma = -\gamma$. Our main result is that the isomorphism class of $S$ is uniquely determined by these periods. A more precise statement will be given in § 4. Details will be published elsewhere.

§ 2. Elliptic surfaces of index 2. It is known that an Enriques surface $S$ has a structure of an elliptic surface (see [1], [8]). That is, there exists a surjective holomorphic map $g: S \to \mathbb{P}^1$ whose general fibre $C$ is an elliptic curve. Moreover there exists a divisor $G$ on $S$ with $CG = 2$. From Kodaira’s formula for the canonical bundles of elliptic surfaces ([6], p. 772), it follows that $g$ has two multiple fibres, both being of multiplicity 2. We view $S$ as an elliptic curve over the function field of $\mathbb{P}^1$. Then $G$ is a divisor of degree 2 on this curve. Hence $G$ defines a rational map $f_1$ of degree 2 of $S$ onto a rational ruled surface $W_1$. This map induces, for each generic fibre $C$, a double covering $C \to \mathbb{P}^1$ which is ramified at 4 points. Let $B_1$ be the branch
locus of \( f \). By applying elementary transformations defined in [3] we can modify \( f \) into a similar rational map \( f': S \to W \) onto a rational ruled surface \( W \) such that its branch locus \( B \) has only the following singularities:

(i) At most a simple triple point, that is, without infinitely near triple points (see [2]).

(ii) \( B \) contains a fibre \( \Gamma \) and \( B = B - \Gamma \) has a double point \( s \) on \( \Gamma \) which, on performing a quadratic transformation at \( s \), gives a double point of its proper transform on the proper transform of \( \Gamma \).

A singularity of the second type corresponds to a double fibre. Hence in our case \( B \) has exactly two of them. Here we remark that the choice of \( W \) is not unique because, if we apply an elementary transformation at \( s \) as in (ii), we obtain another ruled surface and the new branch locus still satisfies the above condition. Because of this phenomenon, we may always take \( P^1 \times P^1 \) as \( W \). In this way we obtain a birational model of \( S \) which is a double covering of \( P^1 \times P^1 \). This model is closely related to the model studied in [1] and [8], which is a double covering of \( P^2 \).

§ 3. Two propositions. The construction in § 2 proves the following

**Proposition 1.** Any two Enriques surfaces are deformation to each other.

To state the second proposition we recall that, if \( T \) is a K3 surface, \( H_4(T, \mathbb{Z}) \) is an even unimodular euclidean lattice of signature (3, 19). Hence it is isomorphic to

\[
A = U_1 \oplus U_2 \oplus U_3 \oplus E_8 \oplus E_8',
\]

where \( U_i = \mathbb{Z} x_i + \mathbb{Z} y_i \) with \( x_i y_i = 1 \), \( x_i^2 = y_i^2 = 0 \) \((i = 1, 2, 3)\) and \( E_8, E_8' \) are copies of the unique even unimodular negative-definite lattice of rank 8. We define an involution \( \rho: A \to A \) by the conditions \( \rho|_{U_i} = -id \), \( \rho(x_i) = x_i \), \( \rho(y_i) = y_i \), \( \rho(E_8) = E_8' \) and that \( \rho \) induces the identity \( E_8 \to E_8' \). We fix \((A, \rho)\) once and for all.

**Proposition 2.** Let \( T \) be the universal covering of an Enriques surface \( S \). Then there exists an isomorphism of euclidean lattices \( \varphi: H_4(T, \mathbb{Z}) \to A \) which satisfies \( \varphi \circ \tau = \rho \circ \varphi \), where \( \tau \) denotes the involution on \( H_4(T, \mathbb{Z}) \) induced by the covering transformation.

By Proposition 1, it suffices to prove Proposition 2 for one special \( S \). On the other hand, if \( g: S \to P^1 \) has a singular fibre of type \( \Pi^* \) (see Kodaira [5], p. 565), i.e., a singular fibre which has the configuration of the extended Dynkin diagram of type \( E_8 \), then it is easy to prove the proposition for \( S \). Therefore the proof is reduced to constructing an Enriques surface with a singular fibre of type \( \Pi^* \). This can be done by using the construction described in § 2.
§ 4. Main Theorem. $(A, \rho)$ being as above, we let $A(-1)$ denote the $(-1)$-eigenspace of $\rho$. Then $A(-1)$ is a euclidean lattice of signature $(2, 10)$ with determinant $2^{10}$. Let $S$ be an Enriques surface and let $\varphi : H_2(T, \mathbb{Z}) \to A$ be as in Proposition 2. Then the integrals (1) determines, via $\varphi$, a linear map $\omega : A(-1) \to \mathbb{C}$ which satisfies the Riemann bilinear relation (see [7]). Hence $\omega$ can be viewed as a point of an open set $D$ in a quadric in $\mathbb{P}^{10}$. $D$ is a disjoint union of two copies of a 10-dimensional symmetric bounded domain of type IV. We let $\Gamma'$ denote the group of those automorphisms of $A$ which commute with the involution $\rho$. Then $\Gamma'$ induces a group $\Gamma$ of automorphism of $A(-1)$, and $\Gamma$ acts discontinuously on $D$. The image $\lambda(S)$ of $\omega$ on $D/\Gamma'$ is uniquely determined by $S$ and does not depend on the choice of $\varphi$. We call $\lambda(S)$ the period of $S$.

Main Theorem. The isomorphism class of an Enriques surface $S$ is uniquely determined by its period $\lambda(S) \in D/\Gamma'$.

The proof uses the Torelli theorem for K3 surfaces due to Piateckii-Shapiro and Shafarevich [7]. Suppose that two Enriques surfaces $S_1$ and $S_2$ have the same period. Then, if $\pi_i : T_i \to S_i$ ($i = 1, 2$) are the universal coverings, there exists an isomorphism $\varphi : H_2(T_1, \mathbb{Z}) \to H_2(T_2, \mathbb{Z})$ which is compatible with involutions and preserves periods. If $\varphi$ maps effective cycles into effective cycles, then $\varphi$ is induced by a unique isomorphism $\Phi : T_1 \to T_2$, and $\Phi$ is compatible with involutions. Hence $S_1$ and $S_2$ are isomorphic to each other. If $\varphi$ does not preserve effective cycles, then we can compose $\varphi$ with a reflexion

$$\gamma \mapsto \gamma + (\gamma \cdot \pi_i^* e) \pi_i^* e$$

with respect to the class $e$ of a rational curve on $S_2$. Composing $\varphi$ with a finite number of such reflexions we may assume that either $\varphi$ or $-\varphi$ preserves effective cycles, and then we are through as above.

Remarks. 1) From the explicit description of $(A, \rho)$, it follows that $D/\Gamma'$ is connected.

2) It can also be proved that $\Gamma$ is an arithmetic subgroup of $SO(2, 10)$ with respect to the $Q$-structure defined by $A(-1)$.

3) It is very likely that our method in [4] can be applied to study the image of the period map $\lambda$ for Enriques surfaces.

4) Our construction in § 2 also works over any algebraically closed field of characteristic $\neq 2$. Using this construction we can prove that any Enriques surface in characteristic $\neq 2$ can be lifted to characteristic 0.

References


