

133. *The Theory of Two-Dimensional Manifolds in the Projective Space of Four Dimensions.*

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In the theory of two-dimensional manifolds F_2 in R_4 there are generally many facts different from those in R_3 . In 1897 K. Kommerell established the theory in the euclidean space,¹⁾ and the theory in the affine space was developed by C. Burstin and W. Mayer in 1927.²⁾ In this note I will build up this theory in the projective space.³⁾ The following results can be modified very easily so as to fit also to the affine space.

1. Let us represent F_2 in parametrical form by $\xi = \xi(u^1, u^2)$, where ξ denotes a system of homogeneous coordinates x^i ($i=1, 2, \dots, 5$) of a point. We consider at first a differential form

$$G_2^* \equiv G_{ij} du^i du^j = | \xi \xi_1 \xi_2 d\xi_1 d\xi_2 |.$$

which is apparently invariant under the projective transformation group, where

$$\xi_1 = \frac{\partial \xi}{\partial u^1}, \quad \xi_2 = \frac{\partial \xi}{\partial u^2}.$$

By a transformation of parameters $u^i = u^i(\bar{u}^1, \bar{u}^2)$, the forms G_2 and G_4 are transformed as follows:

$$G_2^* = \Delta^2 \bar{G}_2^*, \quad G_{,u} = \Delta^2 \bar{G}_{,ij} \frac{\partial \bar{u}^i}{\partial u^j} \frac{\partial \bar{u}^j}{\partial u^i},$$

where

$$\Delta = \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)}.$$

1) K. Kommerell, Die Krümmung der zweidimensionalen Gebilde im ebenen Raume von vier Dimensionen, Dissertation, Tübingen, 1897; Riemannsche Flächen im ebenen Raume von vier Dimensionen, Math. Ann., **60** (1905).

2) C. Burstin and W. Mayer, Über affine Geometrie XVI. Die Geometrie zweifach ausgedehnter Mannigfaltigkeiten F_2 im affinen R_4 , Math. Zeits., **26** (1927), p. 371-407.

3) G. Fubini developed also this theory in another method. G. Fubini—E. Čech, Geometria proiettiva differenziale, Bologna, 1927, p. 631-649.

Whence we can readily see, that the differential form

$$G_2 \equiv g_{ij} du^i du^j = G_2^* (G_{11} G_{22} - G_{12}^2)^{-\frac{1}{3}}$$

is independent of the choice of parameters. We assume that the determinant G of this form is positive; then g_{ij} is considered as a fundamental covariant tensor, from which the fundamental contravariant tensor g^{ij} is determined. Then a linear relation

$$(1) \quad g^{jk} \xi_{jk} + 2p^i \xi_i + q \xi = 0$$

holds between the six vectors ξ , ξ_i and ξ_{jk} , where ξ_{jk} are covariant derivatives of the second order of ξ and

$$p^1 = \frac{1}{2G^2} | \xi \ \xi_2 \ \xi_{11} \ \xi_{12} \ \xi_{22} |, \quad p^2 = -\frac{1}{2G^2} | \xi \ \xi_1 \ \xi_{11} \ \xi_{12} \ \xi_{22} |,$$

$$q = -\frac{1}{G^2} | \xi_1 \ \xi_2 \ \xi_{11} \ \xi_{12} \ \xi_{22} |.$$

2. As the proportional factor of ξ is in general arbitrary, we must normalize this factor invariantly for projective transformations. The quantity

$$I = -q + p^k_{,k} + g_{jk} p^j p^k \quad 3)$$

is multiplied by a factor $\lambda^{-\frac{5}{3}}$ corresponding to the change of proportional factor of ξ : $\bar{\xi} = \lambda \xi$ and is invariant for projective transformations and for any choice of parameters. Let us normalize, therefore, the coordinates ξ such that $I=1$, i.e. $\lambda = I^{-\frac{5}{3}}$, when I is not identically zero.

3. Let $\bar{\xi}$ be the normalized coordinates and put

$$(2) \quad p = g^{jk} \bar{\xi}_{jk} = -2p^i \bar{\xi}_i - q \bar{\xi},$$

$$(3) \quad \eta_{rs} = \bar{\xi}_{rs} - \frac{1}{2} g_{rs} p,$$

then the relation $g^{rs} \eta_{rs} = 0$ follows, and we can prove that there are no more linear relations between η_{rs} . We assume $p^i \neq 0$ and introduce the quantities

$$e^i_1 = \frac{p^i}{p}, \quad e^i_2 = \frac{e^{i,j}_1 e^j}{l}, \quad l = (g_{ij} e^i_{,k} e^k e^j_{,l} e^l)^{\frac{1}{2}}$$

1) I is identically equal to the sum of the so-called invariants h , k of the linear partial differential equation (1).

when $p = (g_{ij}p^i p^j)^{\frac{1}{2}} \neq 0$. Then e^i and e^i_2 are the orthogonal unit vectors with regard to g_{ij} . When $p=0$, we consider another zero-direction \bar{p}^i of the form G_2 , i.e. $g_{ij}\bar{p}^i \bar{p}^j = 0$ and determine \bar{p}^i by the relation $g_{ij}p^i \bar{p}^j = 1$. Then.

$$e^i_1 = \frac{1}{\sqrt{2}}(p^i + \bar{p}^i), \quad e^i_2 = \frac{1}{\sqrt{2}}(p^i - \bar{p}^i)$$

are the orthogonal unit vectors. The points

$$h = \eta_{ij} e^i_1 e^j_1, \quad b = \eta_{ij} e^i_1 e^j_2$$

are called *the principal-normal point* and *the binormal point* respectively. And we can put

$$(4) \quad \eta_{ij} = h_{ij} h + b_{ij} b,$$

where h_{ij} and b_{ij} can be represented by g_{ij} and p^i .

4. From (2), (3) and (4) we get

$$(5) \quad \xi_{ij} = h_{ij} h + b_{ij} b - g_{ij} p^k \xi_k - \frac{1}{2} g_{ij} q \xi.$$

Moreover we put

$$(6) \quad \begin{cases} h_i = h_i h + \bar{h}_i b + (1)\rho_i^k \xi_k + (1)\sigma_i \xi, \\ b_i = \bar{b}_i h + b_i b + (2)\rho_i^k \xi_k + (2)\sigma_i \xi. \end{cases}$$

For the integrability of the differential equations (5) and (6) the so-called conditions of integrability must be satisfied :

$$\xi_{i(jk)} = \frac{1}{2} K_{jki}^l \xi_l,$$

$$h_{(lm)} = 0, \quad b_{(lm)} = 0,$$

where K_{jki}^l is Riemann's curvature tensor. We see from some of the conditions of integrability that the quantities \bar{h}_k , \bar{b}_k , $(2)\rho_k^m$, $(2)\sigma_k$ can be expressed by the other quantities. Furthermore we can prove

$$h_k + b_k = g_{ik} p^i.$$

Therefore we get the theorem :

When the system of the quantities g_{ij} , p^i , h_k , $(1)\rho_k^m$, $(1)\sigma_k$ are given, among which the conditions of integrability hold good, then F_2 is uniquely determined, except for projective transformations.

5. Now we will modify the definition of the evolute in the theory of curves. In general any two two-dimensional planes intersect at only one point. Let us, therefore, represent the unique point of intersection of two consecutive normal planes¹⁾ by ξ^* , then we have

$$(7) \quad \xi^* = \delta_{(12)} P_{tm} \xi + {}_{(2)}P_{tm} \eta + {}_{(1)}P_{tm} \bar{b} du' du^m,$$

where

$$\begin{aligned} {}_{(1)}P_{tm} &= \sqrt{G} \{ {}_{(1)}\rho_{(1)}^2 \delta_m^1 - {}_{(1)}\rho_{(1)}^1 \delta_m^2 \}, \\ {}_{(2)}P_{tm} &= \sqrt{G} \{ {}_{(2)}\rho_{(2)}^1 \delta_m^2 - {}_{(2)}\rho_{(2)}^2 \delta_m^1 \}, \\ {}_{(12)}P_{tm} &= \sqrt{G} \{ {}_{(1)}\rho_{(12)}^1 \delta_m^2 - {}_{(1)}\rho_{(12)}^2 \delta_m^1 \}, \end{aligned}$$

and $\delta_i^k = 0$ for $k \neq i$, $= 1$ for $k = i$. Hence the point ξ^* must describe a conic, if we fix the values of u^i and vary the radio $du^1 : du^2$. We shall call this conic *E-conic*.

The straight lines joining ξ and ξ^* envelope the curve $\mathfrak{R}(\xi^*)$, when and only when the point ξ moves along a curve \mathfrak{R}^1 on F_2 , for which the differential equation.

$$\begin{aligned} &2({}_{(2)}P_{tm} {}_{(1)}P_{np} - {}_{(1)}P_{tm} {}_{(2)}P_{np}) du' du^m du^n \nabla_k (du^p) du^k \\ &+ \begin{vmatrix} {}_{(2)}P_{tm} & {}_{(2)}P_{np,i} + {}_{(2)}P_{np} \bar{h}_i + {}_{(1)}P_{np} \bar{b}_i \\ {}_{(1)}P_{tm} & {}_{(1)}P_{np,i} + {}_{(2)}P_{np} \bar{h}_i + {}_{(1)}P_{np} b_i \end{vmatrix} du^i du' du^m du^n du^p = 0 \end{aligned}$$

holds good, where $\nabla_k (du^p)$ is the covariant derivative of du^p . \mathfrak{R} and \mathfrak{R}^1 are called *evolute* and *E-line* respectively.

The points on the E-conic correspond one-to-one to the tangential directions at the corresponding point on F_2 . Therefore we can define covariant directions by covariant points on the E-conic. For example, to the points of intersection of the principal normal (ξ, η) with the E-conic correspond the *H-directions*, which are given by

$${}_{(1)}P_{tm} du' du^m = 0;$$

to that of the binormal (ξ, \bar{b}) with the E-conic the *B-directions*, which are given by

$${}_{(2)}P_{tm} du' du^m = 0;$$

etc.

1) Normal plane means two-dimensional plane, which determined by three points ξ, η and \bar{b} .

6. Defining the geodesic lines as solutions of the problem of the calculus of variation

$$\delta \int \sqrt{g_{ij} du^i du^j} = 0,$$

we have the differential equations

$$\frac{d^2 u^h}{ds^2} + \left\{ \begin{matrix} ij \\ h \end{matrix} \right\} \frac{du^i}{ds} \frac{du^j}{ds} = 0, \quad ds = (g_{ij} du^i du^j)^{\frac{1}{2}}$$

for the geodesic line u^i . The curvature-vector k^i of a curve u^i is defined as follows:

$$k^i = \nabla_j \left(\frac{du^i}{ds} \right) \frac{du^j}{ds}$$

and the quantity $k = (g_{ij} k^i k^j)^{\frac{1}{2}}$ is the geodesic curvature of the curve. Then we can get the following results:

The form p^i gives us at every point on F_2 such direction that the curve of normal section of this direction has the geodesic curvature zero at that point.

There are two directions such that the geodesic curvature of the curve of normal section has extreme values, and these directions are those of p^i and of its curvature-vector.

7. Let us consider a curve of normal section $\bar{x}(x^1, x^2, x^3, x^4)$ and normalize \bar{x} so that

$$\left| \bar{x} \frac{d\bar{x}}{ds} \frac{d^2 \bar{x}}{ds^2} \frac{d^3 \bar{x}}{ds^3} \right| = 1,$$

then the invariant

$$K = \left| \bar{x} \frac{d\bar{x}}{ds} \frac{d^3 \bar{x}}{ds^3} \frac{d^4 \bar{x}}{ds^4} \right|$$

is called *the curvature of the normal section*. There are 22 directions, for which K has extreme values. For five of these directions K becomes infinitely great, while for the other directions K has a finite value, which is called the principal curvature. And the curves, along which K has always extreme value, are called *lines of curvature*. For other invariants the analogous definitions can also be introduced.