100. Differential Geometry of Conics in the Projective Space of Three Dimensions.

II. Differential invariant forms in the theory of a two-parameter family of conics (first report).

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In my previous paper[1] I have built the theory of a one-parameter family of conics in the projective space of three dimensions. In this little note I will discuss the theory of a two-parameter family of conics in the projective space of three dimensions, as a continuation of that paper. This is done by some modifications of my theory of a m-parameter family of hypersurfaces of the second order in the projective space of n dimensions[2] and of Fubini's surface-theory in the projective space[3]. In this first report I will discuss, as a preliminary, the theory of a two-parameter family of conics in the plane, modifying my theory in the n-dimensional space[4].

1. *The differential forms.* A two-parameter family of conics in the plane can be represented by the equations in parametric form

$$\mathfrak{a}=\mathfrak{a}(u^1, u^2),$$

where u^1 and u^2 are two parameters, when we adopt the coordinate-system $\mathfrak{a}$ of the conic in the plane, which has been introduced in my previous paper[5]. We assume $\mathfrak{a}$ so normalized that

$$(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})=1,$$

i.e.

$$\mathfrak{a}=(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}, \bar{\mathfrak{a}})^{-\frac{1}{3}}\bar{\mathfrak{a}}.$$

Let us consider the differential forms:

$$g_{ij}du^idu^j=2(\mathfrak{a}_i, \mathfrak{a}_j, \mathfrak{a})du^idu^j, \tag{1}$$

$$a_{ijk}du^idu^jdu^k=(\mathfrak{a}_i, \mathfrak{a}_j, \mathfrak{a}_k)du^idu^jdu^k, \tag{2}$$

1) Differential geometry of conics in the projective space of three dimensions, I. Fundamental theorem in the theory of a one-parameter family of conics, these Proceedings **4** (1928), 255–258.

2) See my paper, Fundamental forms in the projective differential geometry of m-parametric families of hypersurfaces of the second order in the n-dimensional space, these Proceedings, **3** (1927), 310–314, and Ueber projektive Differentialgeometrie V, which will be published in the Tohoku Mathematical Journal.

3) See G. Fubini-E. Čech, Geometria proiettiva differenziale, I and II, Bologna, 1926–27.

4) *loc. cit.*

5) *loc. cit.*

which are clearly invariant for any projective transformation and for any change of parameters u^1, u^2, where g_{ij} and a_{ijk} are symmetrical quantities (or symmetrical tensor).

Now we introduce such six conics $\mathfrak{x}_\alpha$, $\mathfrak{X}^\beta$ $(\alpha, \beta=1, 2, 3)$ that

$$\mathfrak{x}_\alpha \mathfrak{A}=\mathfrak{x}_\alpha \mathfrak{A}_i=\mathfrak{a}\mathfrak{X}^\beta=\mathfrak{a}_k\mathfrak{X}^\beta=0, \qquad (3)$$
$$\mathfrak{x}_\alpha\mathfrak{X}^\beta=\delta^\beta_\alpha=\begin{cases}1, & \alpha=\beta,\\ 0, & \alpha\neq\beta,\end{cases}$$

where $\mathfrak{A}$ are contravariant coordinates of the conic $\mathfrak{a}$ and $\mathfrak{A}_i$, $\mathfrak{a}_k$ denote the first covariant derivatives of $\mathfrak{A}$, $\mathfrak{a}$ respectively with regard to the quadratic form (1). Then the second covariant derivatives of $\mathfrak{a}$ and $\mathfrak{A}$ are linearly represented by $\mathfrak{a}$, $\mathfrak{a}_l$, $\mathfrak{x}_\alpha$ and $\mathfrak{A}$, $\mathfrak{A}_l$, $\mathfrak{X}^\beta$, i.e.

$$\begin{cases}\mathfrak{a}_{ij}=-g_{ij}\mathfrak{a}-\dfrac{1}{2}a_{ijk}g^{kl}\mathfrak{a}_l+B_{ij}^{\cdot\cdot\alpha}\mathfrak{x}_\alpha,\\[2mm] \mathfrak{A}_{kj}=-g_{kj}\mathfrak{A}+\dfrac{1}{2}a_{ijk}g^{jl}\mathfrak{A}_l+B_{ik\beta}\mathfrak{X}^\beta,\end{cases} \qquad (4)$$

where
$$\mathfrak{a}_{ij}\mathfrak{X}^\beta=B_{ij}^{\cdot\cdot\beta}, \quad \mathfrak{A}_{ij}\mathfrak{x}_\alpha=B_{ij\alpha}.$$

In (4) we get new differential forms:

$$B_{ij}^{\cdot\cdot\alpha}du^i du^j, \quad B_{ik\beta}du^i du^j, \qquad (5)$$

which remain unaltered by every projective transformation and by any change of parameters. Moreover the first covariant derivatives of $\mathfrak{x}_\alpha$ and $\mathfrak{X}^\beta$ are linearly represented by $\mathfrak{a}_k$, $\mathfrak{x}_\gamma$ or $\mathfrak{A}_i$, $\mathfrak{X}^\gamma$:

$$\begin{cases}\mathfrak{x}_{\alpha,i}=-B_{il\alpha}g^{lk}\mathfrak{a}_k+p_{i\alpha}^{\cdot\cdot\gamma}\mathfrak{x}_\gamma,\\ \mathfrak{X}^\beta_{\cdot,k}=-B_{kl}^{\cdot\cdot\beta}g^{li}\mathfrak{A}_i-p_{k\gamma}^{\cdot\cdot\beta}\mathfrak{X}^\gamma,\end{cases} \qquad (6)$$

where we put

$$\mathfrak{x}_\alpha\mathfrak{X}^\beta_{\cdot,k}=-\mathfrak{x}_{\alpha,k}\mathfrak{X}^\beta=p_{k\alpha}^{\cdot\cdot\beta}. \qquad (7)$$

2. *Determination of* $\mathfrak{X}^\beta$. We can now choose $\mathfrak{X}$ arbitrarily, that is we can introduce new conics $\overline{\mathfrak{X}}^\beta$ instead of $\mathfrak{X}^\beta$ such that

$$\overline{\mathfrak{X}}^\alpha=P^\alpha_\beta\mathfrak{X}^\beta, \qquad (8)$$

where the quantities P^α_β are in general arbitrary functions of parameters. Corresponding to this change (8), the forms (5) are linearly transformed as follows:

$$\overline{B}_{ij}^{\cdot\cdot\alpha}=P^\alpha_\beta B_{ij}^{\cdot\cdot\beta}, \quad \overline{B}_{ij\alpha}P^\alpha_\beta=B_{ij\beta}.$$

By this reason we can choose $\mathfrak{X}^{\alpha}$ so that

$$
(9)\qquad \begin{cases} B_{ij}^{\cdot\cdot 1}du^i du^j = g_{ij}du^i du^j, \\ B_{ij}^{\cdot\cdot 2}du^i du^j = \dfrac{1}{J-I^2}\left\{ I g_{ij} - r_{ij} \right\} du^i du^j, \\ B_{ij}^{\cdot\cdot 3}du^i du^j = q_{ij}du^i du^j, \end{cases}
$$

where $r_{ij}du^i du^j$ is an arbitrary form with coefficients not proportional to those of $g_{ij}du^i du^j$ and

$$
(10)\qquad I = g^{ij}r_{ij}, \quad J = g^{ik}g^{jl}r_{ij}r_{kl} = r_{ij}r^{ij}.
$$

The form $q_{ij}du^i du^j$ is such that

$$
(11)\qquad q_{ij}du^i du^j = \lambda \begin{vmatrix} (du^1)^2 & 2du^1du^2 & (du^2)^2 \\ g_{11} & 2g_{12} & g_{22} \\ r_{11} & 2r_{12} & r_{22} \end{vmatrix} \equiv \lambda \bar{q}_{ij}du^i du^j, \quad \frac{1}{\lambda} = \bar{q}_{ij}\bar{q}^{ij},
$$

then it follows from (9) that

$$
(12)\qquad B_{ij}^{\cdot\cdot\alpha}\, B^{ij\beta} = B^{ij\alpha}\, B_{ij}^{\cdot\cdot\beta} = \delta^{\alpha\beta},
$$

and the $B_{ij}^{\cdot\cdot\alpha}\, du^i du^j$ can be expressed by two forms $g_{ij}du^i du^j$ and $r_{ij}du^i du^j$.

3. *Equations of integrability.* By considering inversely (4) and (6) as the differential equations for $\mathfrak{a}$ and $\mathfrak{x}^{\alpha}$, $\mathfrak{X}^{\beta}$, we can determine the coordinates $\mathfrak{a}$ of the family of conics in the above mentioned forms. For the solvability of these equations it is necessary and sufficient that the following relations hold good

$$
(13)\qquad \begin{cases} g_{i[j}\delta_{k]}^{m} + \dfrac{1}{2}a_{i[j}{}^{\cdot\cdot m}{}_{,k]} - \dfrac{1}{4}a^{l}_{\cdot i[j}a_{k]l}^{\cdot\cdot m} + B_{i[jk]}{}^{m} = \dfrac{1}{2}K_{jki}^{\cdot\cdot\cdot m}, \\ g_{i[j}\delta_{k]}^{m} - \dfrac{1}{2}a^{m}_{\cdot i[j,k]} - \dfrac{1}{4}a^{l}_{\cdot i[j}a_{k]l}^{\cdot\cdot m} + B^{m}_{\cdot [kj]i} = \dfrac{1}{2}K_{jki}^{\cdot\cdot\cdot m}, \end{cases}
$$

$$
(14)\qquad \begin{cases} -\dfrac{1}{2}a^{m}_{\cdot i[j}B_{k]m}^{\cdot\cdot\alpha} + B_{i[j}^{\cdot\cdot\alpha}{}_{,k]} + B_{i[j}^{\cdot\cdot\beta}\, p_{k]\beta}^{\cdot\cdot\alpha} = 0, \\ \dfrac{1}{2}a^{m}_{\cdot i[j}B_{k]m\alpha} + B_{i[j|\alpha|,k]} - B_{i[j|\beta|}p_{k]\alpha}^{\cdot\cdot\beta} = 0, \end{cases}
$$

$$
(15)\qquad p_{[i|\alpha|}^{\cdot\cdot\beta}{}_{,j]} - B^{k}_{\cdot [i|\alpha|}B_{j]k}^{\cdot\cdot\beta} + p_{[i|\alpha|}^{\cdot\cdot\gamma}P_{j]\gamma}^{\cdot\cdot\beta},
$$

where
$$
B_{ijkm} = B_{ij}^{\cdot\cdot\alpha}\, B_{km\alpha}
$$

and $K_{jki}^{\cdot\cdot\cdot m}$ is the Gauss' curvature tensor.

From (13) we get

$$(16)\quad \frac{1}{2}B_{jm}{}^{\beta}\,(\delta^{\beta}_{\alpha}\,\delta_k^j - B^{ij\alpha}\,B_{ik}^{\cdot\cdot\,\beta}) = \left\{\frac{1}{2}K_{jkim} - g_{i(j}\delta_{k)m} - \frac{1}{2}a_{mi(j,k)} + \frac{1}{4}a^{l}_{\cdot i(j}a_{k)lm}\right\}B^{ij\alpha}$$

and also from (14)

$$(17)\quad \frac{1}{2}p_{j\beta}^{\cdot\cdot\alpha}\,(\delta^{\beta}_{\gamma}\delta_k^j - B^{ij\gamma}\,B_{jk}^{\cdot\cdot}\ \ ) = \left\{\frac{1}{2}a^{m}_{\cdot i(j}B_{k)m}^{\cdot\cdot\alpha} - B_{i(j}^{\cdot\cdot\alpha}{}_{,k)}\right\}B^{ij\gamma}.$$

Hence it is known by the last result in No. 2 that the quantities $b_{jm\beta}$ and $p_{j\beta}^{\cdot\cdot\alpha}$ are represented by g_{ij}, r_{ij} and a_{ijk}. Therefore we can conclude that only three differential forms $g_{ij}du^idu^j$, $r_{ij}du^idu^j$ and $a_{ijk}du^idu^jdu^k$ are essential with regard to the family.

Now we put

$$(18)\qquad r_{ij} = a_{ikl}a_j^{\cdot kl},$$

then we get the following fundamental theorem, when r_{ij} are not proportional to g_{ij}.

Given two differential forms $g_{ij}du^idu^j$, $a_{ijk}du^idu^jdu^k$, *between which the relations* (13), (14) *and* (15) *hold good, the family of conics with those forms in the plane is uniquely determined, except for projective transformations.*