PAPERS COMMUNICATED

129. *On the Order of the Absolute Value of a Linear Form (Fifth Report).*

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1. Let us consider the function $\varphi_{\alpha,\beta}(t)$, which is the minimum absolute value of $t(\alpha x-y-\beta)$ for the integral values of x and y, where $|x|<t$. In the former reports I treated mainly two problems: the problem of finding the inferior limit of this function, which may be considered as an extension of a problem solved by Minkowski, and that of finding the superior limit of this function, which was proposed by Hardy and Littlewood. The methods, which I used to solve these two problems were different from each other, although they stand on the same idea. This fact is inconvenient for the further discussion in each problem. In this note I wish to establish a new algorithm, which gives us the relation between the two former algorithms. As the first application of this algorithm I will study the nature of the approximate polygon of the first algorithm more precisely and find the best approximation with regard to the nature of the approximate polygon.

2. *The new algorithm.* We take on the xy-plane a system of lattice points, corresponding to the integral values of x and y, and the straight line $L:\alpha x-y-\beta=0$ and $Y:x=0$, whose intersection is supposed to be M. First we construct a parallelogramm containing M in it, whose sides are parallel to L and Y and which contains no lattice point. Then we translate each side until a lattice point comes on it. We construct all the parallelogramms of such character and call them the approximate parallelogramms. The lattice points on the sides of approximate parallelogramms are called the approximate points. We choose a series of the groups of approximate points (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), ..., (A_n, B_n, C_n, D_n), ... and by an affine transformation transform the points A_n, B_n, C_n, D_n into (0,1), (0,0), (−1,0) (−1,1), and let the new positions of L and Y be $L_n:\alpha_n x-y-\beta_n=0$ and $Y_n:\alpha_n' x+y+\beta_n'=0$. By the similar method as in the former reports we can find the sequence (a_n, b_n, c_n, τ_n), which satisfies the following relations:

$$
\left.\begin{aligned}
&a_n=\left[\frac{1-\beta_n}{\alpha_n}\right], \quad b_n=\left[\frac{\beta_n}{\alpha_n}\right]-1, \quad c_n=\left[\frac{1}{\alpha_n}\right], \\
&c_n=a_n+b_n+\tau_n+1, \quad \nu_n=(-1)^{\tau_n}, \\
&\alpha=\frac{1}{a_1+b_1+2}\,\underset{-}{}\,\frac{\nu_1}{a_2+b_2+2}\,\underset{-}{}\,\frac{\nu_2}{a_3+b_3+2}\,\underset{-}{}\cdots\cdots, \\
&\alpha_n=\frac{1}{a_n+b_n+2}\,\underset{-}{}\,\frac{\nu_n}{a_{n+1}+b_{n+1}+2}\,\underset{-}{}\,\frac{\nu_{n+1}}{a_{n+2}+b_{n+2}+2}\,\underset{-}{}\cdots\cdots. \\
&\beta=1-(a_1+\tau_1)\alpha_1-\nu_1(a_2+\tau_2)\alpha_1\alpha_2-\nu_1\nu_2(a_3+\tau_3)\alpha_1\alpha_2\alpha_3-\cdots\cdots, \\
&\beta_n=1-(a_n+\tau_n)\alpha_n-\nu_n(a_{n+1}+\tau_{n+1})\alpha_n\alpha_{n+1}-\cdots\cdots, \\
&\alpha_{n+1}'=-\nu_n(a_n+b_n+2)-\frac{\nu_{n-1}}{a_{n-1}+b_{n-1}+2} \\
&\qquad\qquad\underset{-}{}\,\frac{\nu_{n-2}}{a_{n-2}+b_{n-2}+2}\,\underset{-}{}\cdots\cdots\,\underset{-}{}\,\frac{\nu_1}{a_1+b_1+2}\cdots\cdots, \\
&\beta_{n+1}'=-1-\nu_n(a_n+\tau_n)+\nu_n\nu_{n-1}\frac{a_{n-1}+\tau_{n-1}}{\alpha_n'} \\
&\qquad\qquad-\nu_n\nu_{n-1}\nu_{n-2}\frac{a_{n-2}+\tau_{n-2}}{\alpha_n'\,\alpha_{n-1}'}+-\cdots\cdots.
\end{aligned}\right\} \quad (1).
$$

3. *The nature of the approximate polygon.* We can find, that the formula (1) is very similar to the formula in the case of regular characteristic numbers in the first method. Here we can prove that, if in (1) all the a_n's and b_n's are not zero, then they coincide with the a_{2m+1}'s and b_{2m+1}'s in the first algorithm and the series of the characteristic numbers in the first algorithm is regular. In the case of regular characteristic numbers, the nature of the approximate polygon is very simple. But in the new method, the algorithm holds good even if some of a_i's and b_i's become zero. From this fact we can determine the nature of the approximate polygon, extending the idea of the regular characteristic numbers, and introducing the sides of length zero. The result runs as follows:

Definition. Let Z_1, Z_2, Z_3, Z_4 be the four approximate polygons in the domain I, II, III, IV respectively, in which $x>0$, $\alpha x-y-\beta<0$; $x<0$, $\alpha x-y-\beta<0$; $x<0$, $\alpha x-y-\beta>0$; $x>0$, $\alpha x-y-\beta>0$ respectively. When a side $\mathfrak{a}$ of Z_3 (for example) corresponds to a side $\mathfrak{a}'$ of Z_1, so that they are parallel to each other and have no lattice point between them, we call the sides $\mathfrak{a}$ and $\mathfrak{a}'$ the sides of the first kind—more precisely, the corresponding sides of the first kind. As a limiting case, one of $\mathfrak{a}$ and $\mathfrak{a}'$, or both of them may be of length zero. We say that the sides, which do not satisfy the above condition, are of the second kind. The sides of the second kind, that lie between two pairs of the corresponding sides of the first kind, are called the corresponding sides of the second kind. We define as the length of the side the number of intervals, into which the said side is divided by the intermedially approximate points on it. We say also, that the edge of the approximate polygon is of the first kind, second kind, third kind respectively, according as it is the meeting point of a side of the first kind with a side of the second kind, of two sides of the first kind or of two sides of the second kind.

Theorem A. The length a of a side of the first kind ST and the length b of the corresponding side $S'T'$ are independent of each other and the approximate polygon of the continued fraction of α has a side of length $a+b$, $a+b+1$ or $a+b+2$, which is parallel to ST and $S'T'$. If a and b are not zero, the first lattice points, which lie on the extensions of ST and $S'T'$ are the edges of the corresponding sides of the second kind of length 1. The side of the second kind ST of length a greater than 1 corresponds to the side $S'T'$ of length b and parallel to ST, so that $|a-b|\leqq 2$, and between them lies one and only one straight line with the lattice points on it. Among the lattice points on this straight line there are two lattice points, which may be considered as the corresponding sides of the first kind of the length zero. There is in the approximate polygon of the continued fraction of α a side parallel to ST and of length between a and b.

4. *The problem of the best approximation.* We say that a lattice point (x, y) gives the best approximation, if for all integral values of x' and y', for which $|x'|\leqq|x|$ the following inequality exists:

$$|\alpha x-y-\beta|\leqq|\alpha x'-y'-\beta|. \tag{A}$$

Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, ... be the sides of the first kind in the right-hand side of the Y-axis and let $\mathfrak{a}_1'$, $\mathfrak{a}_2'$, ... the corresponding sides. Let $\mathfrak{b}_1$, $\mathfrak{b}_2$, ...; $\mathfrak{b}_1'$, $\mathfrak{b}_2'$, ... be the sides of the second kind, so that $\mathfrak{b}_n$ and $\mathfrak{b}_n'$ have the edge on the extensions of $\mathfrak{a}_n$ and $\mathfrak{a}_n'$. We must consider thereby a side of the second kind of length greater than 1 as being constructed by many sides which lie on a straight line. Then we have the

Theorem B. (a) The edge of the first kind P which is an end-point of $\mathfrak{a}_n$ gives the best approximation, if

(i) the first side, which is of shorter length than its corresponding side among $\mathfrak{a}_n$, $\mathfrak{a}_n'$, $\mathfrak{a}_{n+1}$, $\mathfrak{a}_{n+1}'$, ..., lies in the same side of L with P, or if

(ii) the condition (i) is satisfied, when we decrease the length of $\mathfrak{a}_n$ by 1 and

(iii) the first side, which is of shorter length than its corresponding side among $\mathfrak{a}_{n-1}$, $\mathfrak{a}_{n-1}'$, $\mathfrak{a}_{n-2}$, $\mathfrak{a}_{n-2}'$, ..., lies in the same side of the Y-axis with P.

(b) The edge of the second kind, which is the meeting point of $\mathfrak{a}_n$ with $\mathfrak{a}_{n-1}$ gives the best approximation, if the condition (i) or (iii) is satisfied.

(c) The edge of the third kind, which is the meeting point of $\mathfrak{b}_{n-1}$ with $\mathfrak{b}_n$ gives the best approximation, if the condition (ii) and the similar one for $\mathfrak{a}_{n-1}$, $\mathfrak{a}_{n-1}'$, $\mathfrak{a}_{n-2}$, $\mathfrak{a}_{n-2}'$, ... are satisfied.

(d) The intermedially approximate point on a side $\mathfrak{a}$ of the first kind gives the best approximation under the same condition as (a).

(e) The intermedially approximate point of a side of the second kind (which may be considered as an edge of the third kind) never gives the best approximation.