61. On the Relation between $M(r)$ and the Coefficients of a Power Series.

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The relations between the maximum modulus $M(r)=\underset{|z|=r}{\mathrm{Max}}|f(z)|$ of a power series

$$f(z)=a_0+a_1z+a_2z^2+\cdots\cdots$$

and the order of $|a_n|$ are investigated by many authors, in the case of integral transcendental functions, and some analogous results are obtained in the case of a power series with the convergence radius 1. Dr. Beuermann[1] has recently treated the latter case and given the following result.

If we denote

$$\limsup_{r\to 1-0}\frac{\log\log M(r)}{\log\dfrac{1}{1-r}}=\mu, \qquad \limsup_{n\to\infty}\frac{\log\log|a_n|}{\log n}=\sigma \quad (0<\sigma<1),$$

then there exists the relation

$$\mu=\sigma/(1-\sigma).$$

I will here add the following remark.

Theorem. Let

$$\limsup_{r\to 1-0}\frac{\log M(r)}{(1-r)^{-\mu}}=\kappa, \qquad \limsup_{n\to\infty}\frac{\log|a_n|}{n^{\alpha}}=\beta,$$

$$(\mu>0,\ \ \kappa,\ \beta\ \text{finite}\neq 0,\ \ 0<\alpha<1),$$

then $$\mu=\alpha/(1-\alpha), \qquad \kappa=\beta^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}.$$

The method is not essentially new; it is only an application of Laplace's method concerning the functions of large numbers.

Let $$\limsup_{n\to\infty}\frac{\log|a_n|}{n^{\alpha}}=\beta \qquad (0<\alpha<1) \tag{1}$$

be finite. Then for any $\varepsilon>0$, we can determine m such that

$$\frac{\log|a_n|}{n^{\alpha}}<\beta+\varepsilon=\gamma,$$

i.e. $$|a_n|<e^{\gamma n^{\alpha}} \qquad \text{for} \qquad n\geqq m,$$

1) Beuermann, Math. Zeits. **33** (1931).

consequently there exists a constant A such that

$$|a_n| < Ae^{\gamma n^{\alpha}} \qquad \text{for all } n.$$

Then we have

$$M(r) \leqq \sum_0^\infty |a_n| r^n < A \sum_0^\infty e^{\gamma n^{\alpha}} r^n,$$

or putting

$$\log \frac{1}{r} = \tau,$$

$$M(r) < A \sum_0^\infty \exp(\gamma n^\alpha - n\tau).$$

The infinite series on the right hand side has the same order as

$$J(\tau) = \int_0^\infty \exp(\gamma x^\alpha - \tau x) dx,$$

when $r \to 1-0$, $\tau \to 0$. The order of $J(\tau)$ can be determined by Laplace's method in the following way.[1]

Putting $x = \tau^{-\lambda} t$, $\lambda = \frac{1}{1-\alpha}$, we have

$$J(\tau) = \tau^\lambda \int_0^\infty \exp(-\tau^{-\alpha\lambda}(t - \gamma t^\alpha)) dt.$$

Since the function $f(t) = t - \gamma t^\alpha$ takes minimum value $-\gamma^\lambda (1-\alpha)\alpha^{\alpha\lambda}$ at the point $x = \xi = (\gamma\alpha)^\lambda$, and $f''(\xi) = (1-\alpha)(\gamma\alpha)^{-\lambda}$, we have

$$J(\tau) \sim \tau^{-\lambda} \sqrt{2\pi} \exp(-\tau^{-\alpha\lambda} f(\xi)) (\tau^{-\alpha\lambda} f''(\xi))^{-\frac{1}{2}}$$

$$\sim \sqrt{2\pi}\, K^{\frac{1}{2}} \tau^{-\frac{1}{2}(2-\alpha)\lambda} \exp(\tau^{-\alpha\lambda} \gamma^\lambda (1-\alpha)\alpha^{\alpha\lambda}),$$

where

$$K = (\lambda(\gamma\alpha)^\lambda)^{\frac{1}{2}}.$$

If we notice that $\tau = \log \frac{1}{r} \sim 1 - r$, we have

$$\frac{\log M(r)}{(1-r)^{-\alpha\lambda}} < \gamma^\lambda (1-\alpha)\alpha^{\alpha\lambda} + o(1).$$

Therefore we get

$$\limsup_{r \to 1-0} M(r)(1-r)^{\alpha\lambda} \leqq \gamma^\lambda (1-\alpha)\alpha^{\alpha\lambda},$$

consequently, letting $\varepsilon \to 0$, we have

$$\limsup_{r \to 1-0} M(r)(1-r)^{\alpha\lambda} \leqq \beta^\lambda (1-\alpha)\alpha^{\alpha\lambda}. \tag{2}$$

Next let $\beta - \varepsilon = \delta$. Then from the assumption (1), there exists a sequence of integers

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1) The case $\gamma = \frac{1}{\alpha}$ is treated in Polyà-Szegö, Aufgaben und Lehrsätze aus der Analysis, 2, p. 7, Aufgabe 45.

$$n_1 < n_2 < n_3 < \ldots\ldots, \qquad n_\nu \to \infty,$$

such that $\qquad |a_n| > e^{\delta n^\alpha} \qquad$ for $\qquad n = n_1, n_2, \ldots\ldots.$

Denoting now $\quad \tau_i = \delta\, a\, n_i^{\alpha-1} = \log \frac{1}{r_i}, \qquad i = 1, 2, 3, \ldots\ldots$

so that $n_i = (\delta a)^\lambda \tau_i^{-\lambda}$, we get for $n = n_i$, $\tau = \tau_i$, $r = r_i$, $i = 1, 2, \ldots\ldots$

$$\begin{aligned} M(r) &\geqq |a_n| r^n = |a_n| e^{-\tau n} > \exp(\delta n^\alpha - \tau n) \\ &> \exp(\delta(\delta a)^\lambda \tau^{-\alpha\lambda} - (\delta a)^\lambda \tau \cdot \tau^{-\lambda}) \\ &= \exp(\tau^{-\alpha\lambda}(\delta(\delta a)^{\alpha\lambda} - (\delta a)^\lambda)). \end{aligned}$$

Whence follows that

$$\frac{\log M(r)}{(1-r)^{-\alpha\lambda}} > \delta^\lambda (1-a) a^{\alpha\lambda} \quad \text{for} \quad \tau = \tau_1, \tau_2, \ldots\ldots, \quad r = r_1, r_2, \ldots\ldots$$

or, letting $\varepsilon \to 0$,

$$\limsup_{r \to 1-0} \log M(r) \cdot (1-r)^{\alpha\lambda} \geqq \beta^\lambda (1-a) a^{\alpha\lambda}. \tag{3}$$

From (2) and (3) it results

$$\limsup_{r \to 1-0} \log M(r)(1-r)^{\alpha\lambda} = \beta^\lambda (1-a) a^{\alpha\lambda},$$

whence follows the theorem immediately.

By the similar method, we can prove the following

Theorem. If

$$\limsup_{r \to 1-0} \frac{\log M(r)}{(1-r)^{-1}} = \omega, \qquad \limsup_{n \to \infty} \frac{\log |a_n|}{\log n} = \mu$$

be both finite, then $\qquad \mu \leqq \omega \leqq \mu + 1.$

If $\lim\limits_{n \to \infty} \frac{\log |a_n|}{\log n}$ *exists and is equal to* μ, *then* $\mu + \frac{1}{2} \leqq \omega \leqq \mu + 1.$

Suppose $\limsup\limits_{n \to \infty} \frac{\log |a_n|}{\log n} = \mu$, then for any $\varepsilon > 0$ we can determine m such that $\qquad \log |a_n| < (\mu + \varepsilon) \log n,$

i.e. $\qquad |a_n| < n^{\mu+\varepsilon} \qquad$ for $\qquad n \geqq m.$

Therefore there exists a constant A such that

$$|a_n| < A n^{\mu+\varepsilon} \qquad \text{for all } n.$$

Then $\qquad M(r) \leqq \sum_0^\infty |a_n| r^n < A \sum_0^\infty n^{\mu+\varepsilon} r^n \sim A\Gamma(\mu + \varepsilon + 1)(1-r)^{-\mu-\varepsilon-1}.$

Whence follows immediately

$$\limsup_{r \to 1-0} \frac{\log M(r)}{(1-r)^{-1}} \leqq \mu + \varepsilon + 1,$$

consequently $$\limsup_{r\to 1-0} \frac{\log M(r)}{(1-r)^{-1}} \leqq \mu+1 .$$

Next, there exists a sequence of integers $n_1, n_2, n_3, \ldots\ldots \to \infty$ such that $\log|a_n| > (\mu-\varepsilon)\log n$ for $n=n_1, n_2, n_3, \ldots\ldots$

If we denote $r_i=1-\frac{1}{n_i}$, $n_i=\frac{1}{1-r_i}$, then $r_i\to 1-0$ as $n_i\to\infty$,

so that $$M(r)\geqq |a_n|r^n > \left(1-\frac{1}{n}\right)^n n^{\mu-\varepsilon} > \frac{1}{4}(1-r)^{-\mu+\varepsilon}$$

for $n=n_i$, $r=r_i$, $i=1, 2, 3, \ldots\ldots$

Therefore $$\limsup_{r\to 1-0} \frac{\log M(r)}{(1-r)^{-1}} \geqq \mu .$$

If $\lim_{n\to\infty} \frac{\log|a_n|}{\log n}=\mu$ exists, instead of lim sup., then for any $\varepsilon>0$ we can determine m such that

$$\log|a_n| > (\mu-\varepsilon)\log n \qquad \text{for} \qquad n\geqq m .$$

Therefore $$M(r)^2 \geqq \sum_0^\infty |a_n|^2 r^{2n} \geqq \sum_m^\infty |a_n|^2 r^{2n} ,$$

and by putting $r=1-\frac{1}{m}$, $m=\frac{1}{1-r}$,

$$M(r)^2 \geqq m^{2(\mu-\varepsilon)}(r^{2m}+r^{2m+2}+\cdots\cdots) = (1-r)^{-2(\mu-\varepsilon)} r^{2m}(1-r^2)^{-1}$$
$$= (1-r)^{-2(\mu-\varepsilon)-1}\left(1-\frac{1}{m}\right)^{2m}(1+r)^{-1} > \frac{1}{16}(1-r)^{-2(\mu-\varepsilon)-1} .$$

Whence follows $$\limsup_{r\to 1-0} \frac{\log M(r)}{(1-r)^{-1}} \geqq \mu+\frac{1}{2}+\varepsilon \qquad \text{for any } \varepsilon>0,$$

i.e. $$\limsup_{r\to 1-0} \frac{\log M(r)}{(1-r)^{-1}} \geqq \mu+\frac{1}{2}, \qquad \text{q.e.d.}$$