PAPERS COMMUNICATED

77. *On the Starshaped Mapping by an Analytic Function.*

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1. Our object is to prove the following

Theorem. *Let*

$$f(z)=z+a_2z^2+\cdots\cdots+a_nz^n+\cdots\cdots$$

be regular for $|z|<R$ *and* $|f'(z)|<M$ *for* $|z|<R$. *Then the circle* $|z|<\frac{R}{M}$ *is mapped on a starshaped domain with respect to the origin by* $f(z)$ *and also by all its polynomial sections*

$$f_n(z)=z+a_2z^2+\cdots\cdots+a_nz^n \qquad (n=1, 2, \ldots\ldots).$$

Moreover the limiting case is attained by the function

$$f(z)=MR\Big(\frac{M}{R}z+(M^2-1)\log\Big(1-\frac{z}{MR}\Big)\Big).$$

This is a more precise form of a theorem due to S. Takahashi.[1]

2. First we will enunciate a lemma, which is of some interest.

Lemma. *Let* $f(z)=z+a_2z^2+\cdots\cdots+a_nz^n+\cdots\cdots$ *be regular in the unit circle. If* $\sum_2^\infty n|a_n|r^{n-1}<1$, $0<r<1$, *the circle* $|z|\leqq r$ *is mapped on a starshaped domain with respect to the origin by* $f(z)$ *and also by every section* $f_n(z)$.

It is known that $f(z)$ and every section $f_n(z)$ are univalent (schlicht) for $|z|\leqq r$.[2] Therefore $z\frac{f'(z)}{f(z)}$ and $z\frac{f_n'(z)}{f_n(z)}$ $(n=1, 2, \ldots\ldots)$ are regular for $|z|\leqq r$. For the proof it is sufficient to show that

$$R\Big[z\frac{f'(z)}{f(z)}\Big]>0 \quad \text{and} \quad R\Big[z\frac{f_n'(z)}{f_n(z)}\Big]>0^{3)} \quad \text{for} \quad |z|=r.$$

1) S. Takahashi: Tôhoku Math. Journ., **33** (1930), p. 55–60. T. Tannaka: Tôhoku Math. Journ., **35** (1932), p. 43–46. S. Kakeya: Sci. Rep. of Tokyo Bunrika Daigaku, Sect. A, **1** (1932), p. 238–240.

2) T. Itihara: Jap. Journ. of Math., Vol. **6** (1929). See p. 183–184.

3) $R[\zeta]$ denotes the real part of ζ.

We easily see that

$$z\frac{f'(z)}{f(z)}=1+\frac{a_2z+2a_3z^2+\cdots\cdots+(n-1)a_nz^{n-1}+\cdots\cdots}{1+a_2z+a_3z^2+\cdots\cdots+a_nz^{n-1}+\cdots\cdots}.$$

From the assumption $\sum_2^\infty n|a_n|r^{n-1}<1,\ 0<r<1,$ we obtain

$$|a_2|r+2|a_3|r^2+\cdots\cdots+(n-1)|a_n|r^{n-1}+\cdots\cdots$$
$$<1-(|a_2|r+|a_3|r^2+\cdots\cdots+|a_n|r^{n-1}+\cdots\cdots).$$

Consequently, if we put $z\frac{f'(z)}{f(z)}=1+\varphi(z)$, we have, for $|z|=r$,

$$|\varphi(z)|\leqq\frac{|a_2|r+2|a_3|r^2+\cdots\cdots+(n-1)|a_n|r^{n-1}+\cdots\cdots}{1-|a_2|r-|a_3|r^2-\cdots\cdots-|a_n|r^{n-1}-\cdots\cdots}<1.$$

A similar discussion gives, putting $z\frac{f_n'(z)}{f_n(z)}=1+\varphi_n(z)$, $|\varphi_n(z)|<1$, for $|z|=r$. Thus our lemma is proved.

Remark. We easily see that by this lemma we can get more precise forms of some theorems of T. Itihara[1] and A. Kobori.[2]

3. Proof of the theorem. Without loss of generality, we take $R=1$.[3] Since $f(z)$ is regular and $|f'(z)|<M$ for $|z|<1$, Gutzmer's inequality gives

$$1^2+2^2|a_2|^2+3^2|a_3|^2+\cdots\cdots+n^2|a_n|^2+\cdots\cdots\leqq M^2.$$

Therefore

$$\sum_2^\infty n|a_n|r^{n-1}\leqq\sqrt{\sum_2^\infty n^2|a_n|^2}\sqrt{\sum_2^\infty r^{2n-2}}$$
$$\leqq\sqrt{M^2-1}\sqrt{\frac{r^2}{1-r^2}}<1,\quad\text{if}\quad r<\frac{1}{M}.$$

By the lemma our assertion, except the last part, is proved. Considering the function

$$f(z)=M\int_0^z\frac{1-Mz}{M-z}dz=M\Big(Mz+(M^2-1)\log\Big(1-\frac{z}{M}\Big)\Big),$$

which has a vanishing derivative at $z=\frac{1}{M}$, our theorem is completely proved.

1) T. Itihara: loc. cit. p. 183–187.

2) A. Kobori: Memoires of College of Sci., Kyoto Imp. Univ., Ser. A, Vol. **14** (1931), p. 251–262.

3) Consider $\phi(t)=\frac{f(Rt)}{R}$.

4. By the same method used in paragraph 2, we obtain the following:

Theorem. *Let* $f(z)=z+a_2z^2+\cdots\cdots+a_nz^n+\cdots\cdots$ *be regular in the unit-circle. If* $\sum_2^\infty n^2|a_n|r^{n-1}<1,\ \ 0<r<1,$ *the circle* $|z|\leqq r$ *is mapped on a convex domain by* $f(z)$ *and also by every section* $f_n(z)$.