94. On the Distribution of a-points of Solutions for Linear Differential Equation of the Second Order.

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Consider two differential equations

$$y''=\{F(x)+iG(x)\}y\,, \tag{1}$$

$$Z''=H(x)Z\,, \tag{2}$$

where $F(x)$, $G(x)$ and $H(x)$ are real and continuous functions in the domain $D:\ 0\leqq x\leqq c$, such that $H(x)\leqq F(x)$ in D.

We put $|y(x)|=R(x)$ and $\arg y(x)=\theta(x)$, then $R(x)$ will satisfy the differential equation

$$R''=\{F(x)+\theta'(x)^2\}R\,. \tag{3}$$

Lemma. Let the initial values of $R(x)$, $Z(x)$ be given by

$$\begin{cases} R(0)=Z(0)=|a|\,, & a \text{ being a constant, real or imaginary,} \\ R'(0)=Z'(0)>0\,. \end{cases} \tag{4}$$

If we denote by x_1 and x_2 the next $|a|$—points to $x=0$ in D of $R(x)$ and $Z(x)$ respectively, then we must have

$$x_1\geqq x_2\,. \tag{5}$$

Proof. By (4), $R(x)$ and $Z(x)$ are both $>|a|$ for $0<x<\min(x_1, x_2)$. From (2) and (3) we obtain $\frac{d}{dx}(R'Z-RZ')=(\theta'^2+F-H)RZ$. Integrating from 0 to x and remembering (4), we have

$$\begin{cases} R'(x)Z(x)-R(x)Z'(x)=\int_0^x(\theta'^2+F-H)RZdx \\ \qquad\qquad\qquad\qquad \geqq\int_0^x(\theta'^2+F-H)a^2dx\geqq 0\,, \\ 0\leqq x\leqq \min(x_1, x_2)\,. \end{cases} \tag{6}$$

Therefore we shall have $R(x)\geqq Z(x)\geqq|a|$ for $0\leqq x\leqq\min(x_1, x_2)$. Hence, if it be possible that $x_1<x_2$, we should have $|a|=R(x_1)\geqq Z(x_1)\geqq|a|$, that is $Z(x_1)=|a|$ or $x_1=x_2$. This is a contradiction, and so we must have $x_1\geqq x_2$.

Remark. The equality sign of (5) holds if and only if $\theta'(x)\equiv 0$ and $F(x)\equiv H(x)$ in $[0, x_2]$.

Corollary. If we take $H(x) = -\max\limits_{0 \leqq x \leqq x_1} |F(x)| = F_{x_1}$ we may obtain

$$x_1 \geqq \frac{\pi - 2t_g^{-1}\left(\frac{R(0)}{R'(0)}\sqrt{F_{x_1}}\right)}{\sqrt{F_{x_1}}}\ ^{1)}.$$

Because in this case

$$Z(x) = R(0)\sqrt{1 + \frac{R'(0)^2}{R(0)^2 F_{x_1}}}\sin\left(\sqrt{F_{x_1}}\,x + t_g^{-1}\left(\frac{R(0)}{R'(0)}\sqrt{F_{x_1}}\right)\right)$$

and hence

$$x_2 = \frac{\pi - 2t_g^{-1}\left(\frac{R(0)}{R'(0)}\sqrt{F_{x_1}}\right)}{\sqrt{F_{x_1}}}.$$

From these results we deduce the

Theorem. Let $y(x)$ *be an integral of* (1) *such that* $y(0) = a$, $|y(x)|'_{x=0} = R'(0) > 0$. *If the next a-points to* $x = 0$ *of* $y(x)$ *in* D *be denoted by* x_3, *then we must have*

$$(8)\qquad \begin{cases} x_3 \geqq x_2\,, \\ x_3 \geqq \dfrac{\pi - 2t_g^{-1}\left(\frac{|a|}{R'(0)}\sqrt{F_{x_1}}\right)}{\sqrt{F_{x_1}}}. \end{cases}$$

Proof. Since $x_3 \geqq x_1$, (8) follows immediately from (5) and (7).

Corollary. In particular, when $a = 0$, (8) gives

$$(9)\qquad x_3 = x_1 \geqq \frac{\pi}{\sqrt{F_{x_1}}}\ ^{2)}$$

where the equality sign holds if and only if $F(x) \equiv -F_{x_1}$, $G(x) \equiv 0$ in $[0, x_3]$.

Remark. Combined with the method of zero-free regions of E. Hille[3] the formula (9) will play some useful rôle in zero-point problems for linear differential equations of the 2nd order with imaginary coefficients.

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1) The branch of tg^{-1} is to be taken such that $0 \leqq tg^{-1}\alpha \leqq \frac{\pi}{2}$ if $\alpha \geqq 0$.

2) This idea of the generalisation of Sturm's classical theorem was proposed to Mr. H. Nakano, then the result was obtained by either of us independently, almost at the same time. Compare the succeeding paper of Mr. Nakano.

3) See E. Hille; Zero point broblems for linear differential equations of the second order, Mathematisk Tidsskrift, B, 2 (1927).