3. *A Theorem on Cesàro Summability.*

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Recently Prof. Y. Okada proved the theorem:

Let the series[1] $\sum_{n=0}^{\infty} a_n$ *be summable* (C, a) $(a > -1)$, *with sum* s. *If, for any given non-negative* k,

$$\varliminf (C_n^{(k)} - C_m^{(k)}) \geqq 0 \qquad \left(\frac{n}{m} \to 1, \quad n > m \to +\infty\right) \tag{1}$$

holds, then the series is summable (C, k) *with sum* s, *where*

$$C_n^{(k)} = \frac{S_n^{(k)}}{\binom{n+k}{k}},$$

and

$$S_n^{(k)} = \sum_{\nu=0}^{n} \binom{n-\nu-k}{k} a_\nu .$$

In the present paper, it is aimed to deduce a more general theorem, from Schmidt's theorem which runs as follows:

Let the series be summable by Abel's method with sum s. *If*

$$\varliminf (s_n - s_m) \geqq 0 \qquad \left(\frac{n}{m} \to 1, \quad n > m \to +\infty\right)$$

holds, then the series is convergent with sum s, *where*

$$s_n = \sum_{\nu=0}^{n} a_\nu .$$

Theorem. Let the series $\sum a_n$ *be summable by Abel's method with sum* s. *If for any given non-negative* k, (1) *holds, then the series is summable* (C, k) *with sum* s.

Without loss of generality, we can suppose that $s = 0$. Consider the series

$$\sum_{\nu=0}^{n} (C_\nu^{(k)} - C_{\nu-1}^{(k)}), \qquad (C_{-1}^{(k)} = 0),$$

then

$$\sum_{\nu=0}^{n} (C_\nu^{(k)} - C_{\nu-1}^{(k)}) = C_n^{(k)} .$$

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1) We consider here only real series.

2) Y. Okada: On the converse of the consistency of Cesaro's summability, Tohoku Mathematical Journal, **38** (1933).

We can prove that, if

$$\sum_{n=0}^{\infty} a_n x^n \to 0\,, \qquad \text{as} \quad x \to 1-0\,,$$

then
$$\sum_{n=0}^{\infty} (C_n^{(k)} - C_{n-1}^{(k)}) x^n \to 0\,, \qquad \text{as} \quad x \to 1-0\,.$$

For, by the Schmidt's theorem,

$$\sum_{n=0}^{\infty} (C_n^{(k)} - C_{n-1}^{(k)})$$

is convergent, which means the (C, k) summability of $\sum_{n=0}^{\infty} a_n$.

Now
$$\sum_{n=0}^{\infty} (C_n^{(k)} - C_{n-1}^{(k)}) x^n = (1-x) \sum_{n=0}^{\infty} C_n^{(k)} x^n\,.$$

If we put

$$\begin{aligned} P(x) &= (1-x)\Gamma(1+k) \sum_{n=0}^{\infty} \frac{S_n^{(k)}}{(n+1)(n+2)\cdots\cdots(n+k)} x^n \\ &= (1-x)\Gamma(1+k)\left(\int_0^x dx\right)^k \frac{f(x)}{(1-x)^{k+1}}\,, \end{aligned}$$

where
$$f(x) = \sum_{n=0}^{\infty} a_n x^n\,,$$

it suffices to prove that

$$\lim_{x=1-0} P(x) = 0\,.$$

For any $\varepsilon (> 0)$, there is a δ such that

$$|f(x)| < \varepsilon\,, \qquad \text{for} \quad 1-\delta < x < 1\,,$$

and there is an M such that

$$|f(x)| < M\,, \qquad \text{for} \quad 0 \leqq x < 1\,.$$

We have

$$\begin{aligned} |P(x)| &\leqq (1-x)\Gamma(1+k)\left(\int_0^x dx\right)^k \frac{|f(x)|}{(1-x)^{k+1}} \\ &= (1-x)\Gamma(1+k)\left(\int_0^x dx\right)^{k-1}\left(\int_0^{1-\delta} \frac{|f(x)|}{(1-x)^{k+1}} dx + \int_{1-\delta}^x \frac{|f(x)|}{(1-x)^{k+1}} dx\right) \\ &\leqq (1-x)\Gamma(1+k)\frac{M}{\delta^k} + \varepsilon(1-x)\Gamma(1+k)\left(\int_0^x dx\right)^k \frac{1}{(1-x)^{k+1}}\,. \end{aligned}$$

Letting $x \to 1$, we have

$$\lim_{x=1-0} |P(x)| \leqq \varepsilon\,.$$

As ε is arbitrary, we have

$$\lim_{x=1-0} P(x) = 0\,.$$

Thus the theorem is proved.