39. A New Proof of the Andersen's Theorem.

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1. Let
$$\sum_{n=0}^{\infty} a_n \tag{1}$$
be the given series. We put
$$A_n^{(m)}=\binom{m+n}{n},$$
$$S_n^{(r)}=\sum_{\nu=0}^{n} A_{n-\nu}^{(r-1)} s_\nu ,$$
where $s_\nu=a_0+a_1+\cdots\cdots+a_\nu$.

If the limit of
$$\frac{S_n^{(r)}}{A_n^{(r)}} \tag{2}$$
exists and $=s$, then (1) is said to be (C, r)-summable to sum s, and we write $\sum_{n=0}^{\infty} a_n=s\ (C, r)$. If (2) is bounded, then (1) is said to be (C, r)-bounded, and we write $\sum_{n=0}^{\infty} a_n=o(1)\ (C, r)$.

The object of this paper is to prove the following theorems.

Theorem 1. Let $\sigma>\rho>-1$. *If*
$$\sum_{n=0}^{\infty} a_n=O\ (1)(C, \rho)$$
and
$$\sum_{n=0}^{\infty} a_n=s\ (C, \sigma),$$
then $\sum_{n=0}^{\infty} a_n=s\ (C, \tau)$ *for any* $\tau>\rho$.

Theorem 2. Let $\sigma>\rho>-1$. *If*
$$|S_n^{(\rho)}|<A_n^{(\rho)}$$
and
$$|S_n^{(\sigma)}|<A_n^{(\sigma)},$$
then
$$|S_n^{(\tau)}|<\left(2+\frac{\Gamma(\tau-\rho+1)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma-\rho+1)}+o(1)\right)A_n^{(\tau)} \tag{3}$$
for any $\tau>\rho$.

These theorems are due to Andersen.[1] The constant in (3) seems to be new.

1) Andersen: Studier over Cesàro Summabilitetsmetode, 1921. Cf. Zygmund, Math. Zeits., **25** (1926).

In the following, we will prove Theorem 2. Simply modifying the proof, we get the proof of Theorem 1.

2. In order to prove Theorem 2, we transform it in a convenable form.

We put $$\Delta_*^\alpha v_n = \sum_{\nu=0}^{n} A_\nu^{(-\alpha-1)} v_{n-\nu}.$$

Then $\Delta_*^r(\Delta_*^s v_n) = \Delta_*^{r+s} v_n$ for any real r and s.

If we put $$\alpha = \tau - \rho, \qquad \beta = \sigma - \tau$$

and $$S_n^{(\tau)} = v_n,$$

then $$\Delta_*^\alpha v_n = S_n^{(\rho)}, \qquad \Delta_*^{-\beta} v_n = S_n^{(\sigma)}.$$

Therefore, Theorem 2 becomes

Theorem 3. Let $\alpha > 0$ *and* $\beta > 0$. *If*

$$|\Delta_*^\alpha v_n| < A_n^{(\rho)} \tag{4}$$

and

$$|\Delta_*^{-\beta} v_n| < A_n^{(\sigma)}, \tag{5}$$

then

$$|v_n| < \left(2 + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} + o(1)\right) A_n^{(\tau)}.$$

3. We will now prove Theorem 3.

We have

$$\begin{aligned} A_n^{(\beta)} v_n &= \sum_{\nu=0}^{n} A_\nu^{(\beta-1)} v_n \\ &= \sum_{\nu=1}^{n} A_\nu^{(\beta-1)}(v_n - v_{n-\nu}) + \sum_{\nu=0}^{n} A_\nu^{(\beta-1)} v_{n-\nu} \\ &= \sum_{\nu=1}^{n} A_\nu^{(\beta-1)} \sum_{\mu=0}^{\nu-1} \Delta_* v_{n-\mu} + \sum_{\nu=0}^{n} A_\nu^{(\beta-1)} v_{n-\nu} \\ &= \textstyle\sum_1 + \sum_2, \qquad \text{say.} \end{aligned}$$

$$\begin{aligned} \textstyle\sum_1 &= \sum_{\nu=0}^{n-1} (A_n^{(\beta)} - A_\nu^{(\beta)}) \Delta_* v_{n-\nu} \\ &= \sum_{\nu=0}^{n} (A_n^{(\beta)} - A_\nu^{(\beta)}) \Delta_*^{-\alpha+1}(\Delta_*^\alpha v_{n-\nu}) \\ &= \sum_{\nu=0}^{n} (A_n^{(\beta)} - A_\nu^{(\beta)}) \sum_{\mu=0}^{n-\nu} A_\mu^{(\alpha-2)} \Delta_*^\alpha v_{n-\nu} \\ &= \sum_{\nu=0}^{n} (A_n^{(\beta)} A_\nu^{(\alpha-1)} - A_\nu^{(\alpha+\beta-1)}) \Delta_*^\alpha v_{n-\nu}. \end{aligned}$$

$$\begin{aligned} \textstyle\sum_2 &= \sum_{\nu=0}^{n} A_\nu^{(\beta-1)} \Delta_*^\beta(\Delta_*^{-\beta} v_{n-\nu}) \\ &= \sum_{\nu=0}^{n} A_\nu^{(\beta-1)} \sum_{\mu=0}^{n-\nu} A_\mu^{(-\beta-1)} \Delta_*^{-\beta} v_{n-\nu-\mu} = \Delta_*^{-\beta} v_n. \end{aligned}$$

By (4),

$$|\Sigma_1| \leqq A_n^{(\rho)} \sum_{\nu=0}^{n} (A_n^{(\beta)} A_\nu^{(\alpha-1)} + A_\nu^{(\alpha+\beta-1)})$$
$$= A_n^{(\rho)} (A_n^{(\beta)} A_n^{(\alpha)} + A_n^{(\alpha+\beta)}) .$$

By (5), $$|\Sigma_2| < A_n^{(\sigma)} .$$

Thus the theorem is proved.

4. We can prove the following theorem.

Theorem 4. Let $\sigma > \rho > -1$. *If*

$$S_n^{(\rho)} < A_n^{(\rho)}$$

and

$$S_n^{(\sigma)} < A_n^{(\sigma)},$$

then

$$S_n^{(\tau)} < \left(2 + \frac{\Gamma(\tau-\rho+1)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma-\rho+1)} + o(1)\right) A_n^{(\tau)}$$

for any $\tau > \rho$.