1. Let \( N_0(T) \) be the number of zeros of \( \zeta \left( \frac{1}{2} + it \right) \) for \( 0 < t < T \), then
\[
N_0(T) \geq \frac{T}{\pi e} + o(T),
\]
which is a little better than that obtained by R. Kuzmin.\(^1\)

2. Proof. C. L. Siegel\(^2\) proved that the number of zeros of \( \zeta \left( \frac{1}{2} + it \right) \) depends on those of \( f(\sigma + it) \) for \( \sigma < \frac{1}{2} \), where
\[
f(s) = \int_{0}^{1} \frac{e^{-x^2}}{e^{-x^2}} \, dx (s = \sigma + it),
\]
the path of integration is the line parallel to the line bisecting the first and third quadrants and cutting the real axis in a point lying in \((0, 1)\).

Put
\[
g(s) = e^{-\frac{s+1}{2} x^2} e^{-\pi s} \Gamma \left( \frac{1+s}{2} \right) f(s),
\]
and \( U = T^a \left( \frac{13}{14} < a < 1 \right) \) and \( N(T) \) be the number of zeros of \( f(s) \) for \( s \) lying in the rectangle \(-T^a < \sigma < \frac{1}{2}, \quad T < t < T + U\). By making a detailed calculation as in Siegel's paper,\(^3\) we have
\[
(2.1) \quad N_0(T+U) - N_0(T) \geq 2N(T) + O(T^{\frac{13}{14}}).
\]
By a similar method for calculating the mean value formula of zeta-function we have
\[
\int_{T}^{T+U} |g(\sigma + it)|^2 \, dt = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2 - \sigma} \right) \sqrt{\frac{2}{\pi}} \frac{T^{\frac{1}{2}} U + O(U^2 T^{-\frac{1}{2}}) + O(T^5)}
\]
for \( \sigma < \frac{1}{4} \).

Using a known inequality\(^4\) we have

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3) C. L. Siegel. Loc. cit.
4) Hardy-Littlewood-Pólya, Inequality.
\[
\int_{T}^{T+U} \log |g(\sigma+it)| \, dt \leq \frac{U}{4} \log T + \frac{U}{2} \log \left(1 + \frac{\sqrt{2}}{2(\frac{1}{2} - \sigma)}\right)
+ O(U^2T^{-1}) + O(T^2).
\]

On the other hand
\[
\int_{T}^{T+U} \log \left|g\left(\frac{1}{2}+it\right)\right| \, dt = \frac{U}{4} \log T + \frac{U}{2} \log \sqrt{\frac{2}{\pi}}
+ 2\pi \sum_{a > \frac{1}{2}} \left(a - \frac{1}{2}\right) + O(U^2T^{-1}),
\]
where \(a's\) run over all real parts of zeros of \(f(s)\) for \(\frac{1}{2} < \sigma < 2\) and \(T < t < T+U\). Hence
\[
\left(\frac{1}{2} - \sigma\right)2\pi N(T) \geq \frac{U}{2} \log 2\left(\frac{1}{2} - \sigma\right) + O(U^2T^{-1}) + O(T^{13/2}).
\]

If we take \(\sigma = \frac{1}{2} - \frac{1}{2}\), then
\[
(2.2) \quad 2\pi N(T) \geq \frac{U}{e} + o(U).
\]

By (2.1) and (2.2) we have
\[
N_0(T+U) - N_0(T) \geq \frac{1}{\pi e} U + o(U),
\]
for \(T > t_0\).

Hence
\[
N_0(T) - N_0(T-U) \geq \frac{1}{\pi e} U + o(U),
\]

..............................

\[
N_0(T-x^{-1}U) - N_0(T-xU) \geq \frac{1}{\pi e} U + o(U).
\]

If we take \(x = \left[\frac{T-t_0}{U}\right]\), then \(T-xU \geq t_0\), and
\[
x \geq \frac{T-t_0}{U} - 1.
\]

Hence
\[
N_0(T) - N_0(T-xU) \geq \frac{T}{\pi e} + o(T).
\]

Thus we have the required result.

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1) Siegel, p. 68.