79. *On the Boundary Values of Analytic Functions.*

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I. We consider a simple closed rectifiable curve C on Gaussian plane, and we give a continuous function $f(t)$ on C. If there exists a function $F(z)$ which is analytic within C and is continuous up to the boundary C, satisfying $F(t)=f(t)$ on C, then we must have

$$\int_C f(t)\,t^m dt=0\,, \qquad m=0,1,2,\ldots\ldots \tag{1}$$

since z^m is analytic within and on C.

The equations (1) are also sufficient, under a certain condition, *for the existence of such an analytic function* $F(z)$ as above. For example, (1) is sufficient if $f(t)$ is given to be analytic along C.[1] Also it is sufficient if the curve C is analytic.[2]

The paper is devoted to prove *the sufficiency of (1), in the case where C has the following property P*:

To every point t on C, we can so associate a pair of opposite sectors (the sides of one sector being the elongations of the other's) of center t that 1) it varies continuously with t, 2) its radius ρ and central angle ω $(0<\omega<\pi)$ are fixed, and 3) the one sector lies within C while the other lies without C.

Any curve with *continuous tangent*, for example, evidently possesses the property P. For such curve, we can give previously any angle ω less than π, taking ρ sufficiently small, and make the sectors symmetric with respect to the tangent.

II. For proving the existence of such function $F(z)$ as above, it is sufficient to see that $F(z)$ is analytic within C and tends *uniformly* to $f(t)$ when z tends to t along the bisector of the inner sector corresponding to t. Because any point z within C which approaches to t on C should approach to t_1 on C, which is near to t and the bisector of whose corresponding sectors passes through z. So $F(z)$, approaching $f(t_1)$, will tend to $f(t)$. This is based upon the fact that the said bisector generates continuously the inner side of the curve C.

When the required function $F(z)$ should exist, it must be represented, within C, by Cauchy's integral

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt\,. \tag{2}$$

So we are only to show, under the condition (1), that this integral (2) (which is evidently analytic within C) tends uniformly to $f(t_0)$ when z tends to t_0 along the said bisector at t_0.

1) S. Kakeya, Tohoku Math. Journ., **5** (1914), p. 42.
2) J. L. Walsh, Trans. Amer. Math. Soc., **30** (1928), p. 327.

The integral (2) is also analytic in the outside of C, being zero at infinity. The coefficient of $\frac{1}{z^k}$ $(k=1, 2, 3, \ldots\ldots)$ in the expansion of (2) at infinity is $\frac{-1}{2\pi i}\int_C f(t)t^{k-1}dt$. Hence the condition (1) *is equivalent to that*

$$\int_C \frac{f(t)}{t-z'}dt=0 \tag{3}$$

for all z' in the outside of C.

Under these remarks, we now proceed to the proof.

III. Taking a fixed point t_0 on C, we have

$$\frac{1}{2\pi i}\int_C \frac{f(t_0)}{t-z}dt=f(t_0) \tag{4}$$

and

$$\frac{1}{2\pi i}\int_C \frac{f(t_0)}{t-z'}dt=0 \tag{5}$$

where z and z' denote respectively any points within and without C. So the equation

$$\lim_{z\to t_0}\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt=f(t_0) \tag{6}$$

which we are to prove, under the condition (3), is equivalent to

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt-\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z'}dt$$

$$\to\frac{1}{2\pi i}\int_C \frac{f(t_0)}{t-z}dt-\frac{1}{2\pi i}\int_C \frac{f(t_0)}{t-z'}dt \tag{7}$$

or

$$\int_C \frac{\{f(t)-f(t_0)\}\,(z-z')}{(t-z)\,(t-z')}dt\to 0 \tag{8}$$

uniformly for $z\to t_0$ along the bisector. Here z' may be any point without C. So we take as z' the point which is symmetric for z with respect to t_0. Then $z\to t_0$ is equivalent to $|z-z'|\to 0$.

Denote the length of the arc $\widehat{t_0t}$ by s, and the arc containing t_0, for which $s\leqq\sigma$, by C_0. Take σ so small that

$$|f(t)-f(t_0)|\leqq\varepsilon \qquad \text{on } C_0. \tag{9}$$

Corresponding to this σ, there exists a constant $M(\sigma)$, independent of z, z', such that

$$\frac{1}{|t-z|\,|t-z'|}\leqq M(\sigma) \qquad \text{on } C-C_0. \tag{10}$$

The determination of σ and $M(\sigma)$ can be made independently of the position of t_0.

If we can find a constant N independent of σ, z, z' and t_0 such that

$$\int_{C_0} \frac{|dt|}{|t-z|\,|t-z'|} \leqq \frac{N}{|z-z'|} \tag{11}$$

then we get

$$\left| \int_C \frac{\{f(t)-f(t_0)\}\,(z-z')}{(t-z)\,(t-z')} dt \right| \leqq \left| \int_{C_0} \right| + \left| \int_{C-C_0} \right|$$

$$\leqq \varepsilon N + 2GM(\sigma)S\,|z-z'| \tag{12}$$

where G is the greatest magnitude of $f(t)$ and S is the whole length of C. Since ε can be taken arbitrarily, the inequality (12) shows that (8) holds good.

Hence there remains only to prove the existence of a constant N of (11).

Evidently, we can find such a constant σ_0 that, if $\sigma \leqq \sigma_0$ the arc C_0 lies wholly without the angles of the sectors corresponding to t_0. We may confine ourselves to such a small σ. Let the projection of t on the bisector of the complementary angles of the sectors be t', then there exists a positive constant λ depending on ω such that

$$|t-z| \geqq \lambda\,|t'-z|$$

$$|t-z'| \geqq \lambda\,|t'-z'| = \lambda\,|t'-z| \tag{13}$$

since t lies in the said complementary angles. Hence we have

$$\int_{C_0} \frac{|dt|}{|t-z|\,|t-z'|} \leqq \frac{1}{\lambda^2} \int_{C_0} \frac{|dt|}{|t'-z|^2} = \frac{1}{\lambda^2} \int_{-\sigma}^{\sigma} \frac{ds}{x^2(s)+r^2} \tag{14}$$

where $x(s) = |t_0 - t'|$, $r = |t_0 - z| = \dfrac{|z-z'|}{2}$ and s is measured with the sense.

Changing the variable by

$$s = ru\,, \qquad ds = r\,du \tag{15}$$

we get

$$\int_{-\sigma}^{\sigma} \frac{ds}{x^2+r^2} = \frac{1}{r} \int_{-\frac{\sigma}{r}}^{\frac{\sigma}{r}} \frac{du}{\left\{\dfrac{x(ru)}{r}\right\}^2+1}\,. \tag{16}$$

Since our associated sectors varies continuously with t_0, we see that any chord of C_0 makes such an angle θ, with the line t_0t', that

$$\frac{\pi}{2} - \frac{\omega}{2} + \delta > \theta > -\left(\frac{\pi}{2} - \frac{\omega}{2} + \delta\right) \tag{17}$$

where $\delta(>0)$ can be made so small as $0 < \frac{\pi}{2} - \frac{\omega}{2} + \delta < \frac{\pi}{2}$ by taking

σ_0 suitably small. In this case, we evidently have[1]

$$s \leqq x \sec\left\{\frac{\pi}{2}-\frac{\omega}{2}+\delta\right\}=x\mu \qquad \text{(say, for brevity).} \tag{18}$$

And hence

$$\int_{-\frac{\sigma}{r}}^{\frac{\sigma}{r}} \frac{du}{\left\{\frac{x(ru)}{r}\right\}^2+1} \leqq \int_{-\frac{\sigma}{r}}^{\frac{\sigma}{r}} \frac{du}{\left(\frac{u}{\mu}\right)^2+1} \leqq \int_{-\infty}^{\infty} \frac{du}{\left(\frac{u}{\mu}\right)^2+1}=\mu\pi . \tag{19}$$

Thus we get the inequality (11), by putting

$$N=\frac{\mu\pi}{\lambda^2}, \tag{20}$$

which completes our proof.

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1) Even in the case where the variation of our sectors is not continuous, we can get a certain limitation on the whole length of the curve, if it lies within a given limited domain. See the next paper of Kakeya and Kunugi, in this proceeding.