PAPERS COMMUNICATED

101. *A Theorem Concerning the Fourier Series of a Quadratically Summable Function.*

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1. Recently Mr. R. Salem[1] has proved the following theorem:

If $f(x)$ is a bounded periodic function with period 2π and its Fourier coefficients are a_n, b_n, then the following relation holds for almost all values of x,

$$\lim_{s\to 0}\left[\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\sqrt{\log n}}\right]=f(x). \tag{1}$$

Actually he proved the relation (1) replacing more general sequence $\{\psi_n(s)\}$ for $\{1/(1+s\sqrt{\log n})\}$. The object of the present paper is to prove the validity of (1) under the condition that $f(x)\in L_2$, i.e. is quadratically summable. In this form the theorem says more than the well known theorem of Kolmogoroff-Seliverstoff-Plessner[2] concerning the convergence factor of the Fourier series of a quadratically summable function. But we can prove our theorem by using the theorem of Kolmogoroff-Seliverstoff-Plessner.

2. *Theorem 1. If $f(x)\in L_2$ and is periodic with period 2π and a_n, b_n are its Fourier coefficients, then the relation (1) holds for almost all values of x.*

Theorem 2. In Theorem 1, we can replace the sequence $\{1/(1+s\sqrt{\log n})\}$ by the sequence $\{\psi_n(s)\}$ which satisfies the following conditions:

1°. *$\{\psi_n(s)\}$ is the decreasing and convex sequence of positive functions, $0<s\leqq 1\,(\psi_0(s)=1)$.*

2°. $\lim_{s\to 0}\psi_n(s)=1$, (n fixed).

3°. $\lim_{n\to\infty}\psi_n(s)=0$, ($s$ fixed, >0).

4°. $\psi_n(s)=O\,(\sqrt{\log n})$, ($s$ fixed, >0).

5°. $\psi_n(s)$ has a finite number of maxima for any fixed n.

The proof of Theorem 2 is quite similar as that of Theorem 1 and so we only prove Theorem 1.

Let E_1 be the set of x such that

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1) R. Salem, Sur une méthode de sommation, valable presque partout, pour les séries de Fourier de fonction continue, Comptes Rendus, **205** (1937), pp. 14-16.

" , Sur une généralisation du procédé de sommation de Poisson, ibid., **205** (1937), pp. 311-313.

2) See, Zygmund, Trigonometrical series, Warsaw (1935), pp. 253-255.

$$\sum_{n=2}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{\sqrt{\log n}}$$

converges. Then $mE_1=2\pi$. This is the theorem of Kolmogoroff-Seliverstoff-Plessner. And for $x\in E_1$,

$$\sum_{n=2}^{N}(a_n\cos nx+b_n\sin nx)=o(\sqrt{\log N}).$$

We can easily verify that for $x\in E_1$,

$$f(x,s)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\sqrt{\log n}}$$

converges and its N-th partial sum is $o(\sqrt{\log n})$ for every value of s. The Parseval relation shows that

$$\text{(2)}\qquad \lim_{\to 0}\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x,s)-f(x)|^2dx=\lim_{s\to 0}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\left(\frac{1}{1+s\sqrt{\log n}}-1\right)^2$$
$$=0.$$

From the kown result concerning the convergence in mean, we see that there exists a sequence $\{s_n\}(\lim_{n\to\infty}s_n=0)$ such that

$$\lim_{s\to\infty}f(x,s_n)=f(x)$$

for almost all values of x.

Now let $f(x)=f^+(x)-f^-(x)$, where

$$f^+(x)=f(x),\ \text{if}\ f(x)\geqq 0,\qquad f^-(x)=-f(x),\ \text{if}\ f(x)<0,$$
$$=0,\ \text{otherwise},\qquad =0,\ \text{otherwise}.$$

Then $f^+(x),f^-(x)\geqq 0$ and $f^+(x),f^-(x)\leqq|f(x)|$.

Write

$$f^+(x)\sim\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos nx+\beta_n\sin nx),$$

$$f^-(x)\sim\frac{1}{2}\gamma_0+\sum_{n=1}^{\infty}(\gamma_n\cos nx+\delta_n\sin nx),$$

then clearly $\alpha_n-\gamma_n=a_n$, $\beta_n-\delta_n=b_n$. Similar arguments as above show that there exist a set S_1 and a sequence $\{s_n\}$ such that $mS_1=2\pi$ and for $x\in S_1, f^+(x,s)$ converges and the N-th partial sums are $o(\sqrt{\log N})$ and $\lim_{n\to\infty}f^+(x,s_n)=f^+(x)$. By applying the Abel's transformation twice, we have, if $x\in S_1$

$$\text{(3)}\qquad f^+(x,s)=\lim_{N\to\infty}\left\{\frac{1}{2}\alpha_0+\sum_{n=1}^{N}\frac{\alpha_n\cos nx+\beta_n\sin nx}{1+s\sqrt{\log n}}\right\}$$

$$\text{(4)}\qquad =\lim_{N\to\infty}\left\{\sum_{n=0}^{N-2}K_n(x)\Delta^2\frac{1}{1+s\sqrt{\log n}}\right.$$
$$\left.+K_{N-1}(x)\Delta\frac{1}{1+s\sqrt{\log(N-1)}}+S_N(x)\frac{1}{1+s\sqrt{\log N}}\right\},$$

where $S_n(x)$ is the N-th partial sum of the series in the bracket of the right hand side of (3) and

$$K_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin^2(nt/2)}{\sin^2(t/2)}dt\,, \quad (n>0)\,, \quad K_0(x)=a_0$$

and

$$\Delta a_p = a_p - a_{p+1}\,.$$

In this, we must replace 1 for $\dfrac{1}{1+s\sqrt{\log n}}$ if $n=0$. The last term in the bracket of (4) tends to zero as $N\to\infty$ and the same is also easily verified for the second term. Thus

$$f^+(x,s)=\sum_{n=0}^{\infty} K_n(x)\Delta^2\frac{1}{1+s\sqrt{\log n}}\,,$$

where we notice that $K_n(x)$ and $\Delta^2\dfrac{1}{1+s\sqrt{\log n}}$ are positive and $f^+(x,s)$ is also positive. Now take two numbers s_p, s_{p+1} from $\{s_n\}$ such that $s_{p+1}\leqq s < s_p$. Then we have

$$\begin{aligned}0\leqq f^+(x,s)\leqq \sum_{n=0}^{\infty} K_n(x)\Delta^2\frac{1}{1+s_p\sqrt{\log n}}+\sum_{n=0}^{\infty} K_n(x)\Delta^2\frac{1}{1+s_{p+1}\sqrt{\log n}}\\ +\sum_{n_s-2}^{n_s+2} K_n(x)\Delta^2\frac{1}{1+s\sqrt{\log n}}\\ =f^+(x,s_p)+f^+(x,s_{p+1})+\frac{1}{n_s}\int_0^{2\pi}|f(x+t)|\frac{\sin^2(n_s t/2)}{\sin^2(t/2)}dt\,,\end{aligned}$$

for some n_s which tends to ∞ as $s\to 0$.

Hence we have

$$\overline{\lim_{s\to 0}}\, f^+(x,s)\leqq 2f^+(x)\leqq 2|f(x)|\,.$$

Similarly there exists a set S_2 such that for $x\in S_2$,

$$\overline{\lim_{s\to 0}}\, f^-(x,s)\leqq 3|f(x)|\,.$$

Thus for $x\in S_1\cdot S_2$ we have

$$\overline{\lim_{s\to 0}}\,|f(x,s)|\leqq \overline{\lim_{s\to 0}}\, f^+(x,s)+\overline{\lim_{s\to 0}}\, f^-(x,s)\leqq 6|f(x)|\,.$$

Now let

$$f_M\sim\sum_{n=M+1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

and

$$f_M(x,s)=\sum_{n=M+1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\sqrt{\log n}}\,, \qquad (x\in E_1)\,.$$

Then there exists a set E_M such that $mE_M=2\pi$ and for $x\in E_M$,

$$\overline{\lim_{s\to 0}}\,|f_M(x,s)|\leqq 6\,|f_M(x)|\,. \tag{5}$$

Thus in ΠE_M, $\overline{\lim\limits_{s\to 0}}\,|f_M(x,s)|$ is finite for every M. Squaring and integrating both sides of (5), we have

$$\int_{-\pi}^{\pi}\{\overline{\lim_{s\to 0}}\,|f_M(x,s)|\}^2dx\leqq 6\pi\int_{-\pi}^{\pi}|f_M(x)|^2dx=6\pi\sum_{n=M+1}^{\infty}(a_n^2+b_n^2)\,.$$

Hence we get

$$\lim_{M\to\infty}\int_{-\pi}^{\pi}\{\overline{\lim_{s\to 0}}\,|f_M(x,s)|\}^2dx=0\,.$$

Therefore there exist a set E and a sequence M_k such that $mE=2\pi$, and for $x\in E$

$$\lim_{K\to\infty}\overline{\lim_{s\to 0}}\,|f_{M_k}(x,s)|=0\,.$$

Now for $x\in E$,

$$\begin{aligned}\lim_{s,s'\to 0}|f(x,s)-f(x,s')|&\leqq \lim_{s,s'\to 0}\Big|\sum_{n=1}^{M_k}(a_n\cos nx+b_n\sin nx)\\&\qquad\times\left(\frac{1}{1+s\sqrt{\log n}}-\frac{1}{1+s'\sqrt{\log n}}\right)\\&\quad+2\,\overline{\lim_{s\to 0}}\left|\sum_{n=M_k+1}^{\infty}(a_n\cos nx+b_n\sin nx)\frac{1}{1+s\sqrt{\log n}}\right|\\&=2\,\overline{\lim_{s\to 0}}\left|\sum_{n=M_k+1}^{\infty}(a_n\cos nx+b_n\sin nx)\frac{1}{1+s\sqrt{\log n}}\right|\end{aligned}$$

which is arbitrarily small by taking k large. Thus $\lim\limits_{s\to 0}f(x,s)$ exists for $x\in E$. The fact that the limiting value is $f(x)$ is an immediate consequence of (2). Thus we complete the proof.