PAPERS COMMUNICATED

72. *Asymptotic Expansions in the Heaviside's Operational Calculus.*

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1. Let p denote d/dx and $F(z)$ be a power series $\sum a_n z^n$. Operating $F(p)p^{1/2}$ on 1 we have formally

$$\frac{1}{\sqrt{\pi x}}\left(a_0-\frac{a_1}{(2x)}+\frac{1\cdot 3\, a_2}{(2x)^2}+\cdots(-1)^n\frac{1\cdot 3\cdot 5\ldots(2n-1)a_n}{(2x)^n}+\cdots\right), \tag{1}$$

which is in general divergent. Denoting this series by $S(x)$ and putting

$$F(p)p^{1/2}.\ 1=h(x),$$

we have

$$h(x)\sim S(x),$$

where $\sim$ means that $S(x)$ is an asymptotic expansion of $h(x)$. This is one of the Heaviside's rules.

To give a rigorous basis for this Heaviside's rule, Mr. Carson introduced Laplace operator and defined $h(x)$ by the relation

$$\frac{F(p)}{p^{1/2}}=\int_0^\infty e^{-pt}h(t)\,dt \tag{2}$$

and proceeded as follows.[1)]

He defined $g(t)$ by
$$F(p)=\int_0^\infty e^{-pt}g(t)\,dt$$
and deduced
$$a_n=\frac{(-1)^n}{n!}\int_0^\infty g(t)t^n\,dt \qquad (n=0,1,2,\ldots)$$
by assuming the existence of $\int_0^\infty g(t)t^n\,dt$ and the termwise integrability of $\sum\frac{(-1)^n t^n}{n!}g(t)$. Then by making use of the relation

$$\frac{1}{p^{1/2}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}$$

and the so-called "Faltungssatz," he deduced further

$$h(x)=\int_0^x\frac{g(t)}{\sqrt{\pi(x-t)}}\,dt$$
$$=\frac{1}{\sqrt{\pi x}}\int_0^x g(t)\left(\sum\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!\,(2x)^n}t^n\right)dt.$$

If the termwise integrability be assumed, it results

1) See Carson, Electric circuit theory and the operational calculus, 1926, Chap. V.

$$h(x)=\frac{1}{\sqrt{\pi x}}\sum\frac{1\cdot3\cdot5\ldots(2n-1)}{n!\,(2x)^n}\int_0^x g(t)t^n\,dt.$$

Putting here $\int_0^x=\int_0^\infty-\int_x^\infty$,

$$h(x)=\frac{1}{\sqrt{\pi x}}\sum(-1)^n\frac{1\cdot3\cdot5\ldots(2n-1)}{(2x)^n}a_n$$

$$-\frac{1}{\sqrt{\pi x}}\sum\frac{1\cdot3\cdot5\ldots(2n-1)}{n!\,(2x)^n}\int_x^\infty g(t)t^n\,dt$$

$$=S(x)-\frac{1}{\sqrt{\pi x}}\sum\frac{1\cdot3\cdot5\ldots(2n-1)}{n!\,(2x)^n}\int_x^\infty g(t)t^n\,dt.$$

Mr. Carson denotes this in the form

$$S(x)-\int_x^\infty\frac{g(t)\,dt}{\sqrt{\pi(x-t)}}$$

and states that $S(x)$ is the asymptotic expansion of $h(x)$.

This latter part of his reasonings is, however, not quite rigorous, since in the interval (x,∞), $(x-t)^{-1/2}$ can not be expanded in the form $\sum\frac{1\cdot3\cdot5\ldots(2n-1)}{\sqrt{x}\;n!\,(2x)^n}t^n$, and although $\int_x^\infty\frac{g(t)}{\sqrt{\pi(x-t)}}dt\to0$ as $x\to\infty$, we can not say that $S(x)$ is the asymptotic expansion of $h(x)$, unless

$$\lim_{|x|\to\infty}x^n\int_x^\infty\frac{g(t)\,dt}{\sqrt{\pi(x-t)}}=0\quad\text{for all } n$$

were not proved.

Mr. Carson took further a special case $F(p)=\frac{1}{p^2+\omega^2}$ into consideration, and deduced

$$h(x)=\frac{1}{\omega\sqrt{\pi}}\int_0^x\frac{\sin\omega(x-t)}{\sqrt{t}}\,dt$$

$$=\frac{1}{\omega\sqrt{\pi}}\left\{-\cos\omega x\int_0^x\frac{\sin\omega t}{\sqrt{t}}\,dt+\sin\omega x\int_0^x\frac{\cos\omega t}{\sqrt{t}}\,dt\right\}.$$

Observing the oscillating character of this solution and comparing with the Heaviside's solution $S(x)=\frac{1}{\omega^2\sqrt{\pi x}}\left\{1-\frac{1}{(2\omega x)^2}+\frac{1\cdot3\cdot5\cdot7}{(2\omega x)^4}-\cdots\cdots\right\}$, he considered this example as a proof of the existence of some cases where Heaviside's solution does not give the exact solution.

2. In the following lines we wish to discuss the relation of the Heaviside's solution and that of Carson.

Inverting the relation (2)

we get $$h(x)=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{e^{xz}F(z)}{z^{1/2}}\,dz\,,^{1)}\qquad(3)$$

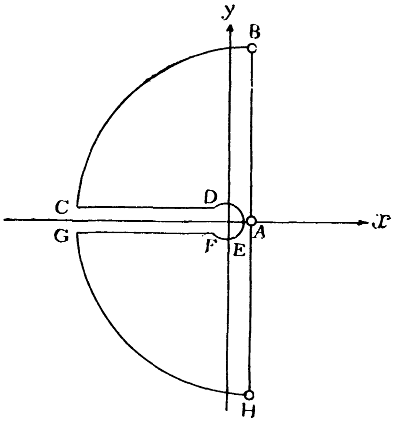
1) This formula was deduced from Fourier double integral formula by Levy, Bull. des sciences math., (2) **50** (1926), and March, Bull. Amer. M. S., **33** (1927). But I have given this formula in my paper, Über Abelsche erzeugende Funktion und Darstellbarkeitsbedingung von Funktionen durch Dirichletsche Reihe, Tohoku Math. Journ., **17** (1920).

if we assume $\left|\frac{F(z)}{z^{1/2}}\right|\to 0$, for $|t|\to\infty$, $(z=c+it)$,

$$\left|\frac{F(z)}{z^{1/2}}\right|\to 0 \text{ uniformly for } \sigma\to\infty, \quad (z=\sigma+it) \tag{a}$$

and $F(z)$ is regular in the half-plane $R(z)\geqq c>0$.

We assume further that $F(z)$ is a meromorphic function whose poles do not lie on the negative real axis, the origin inclusive.



Now let A be the point $z=c$, and $BCGH$ be a semi circle with centre A and radius r. Then the integral of $e^{xz}F(z)/2\pi iz^{1/2}$ along the straight line HAB is equal to the sum of integrals along the circular arcs BC, GH, those along real negative axis CD, FG, and a small circle DEF about the origin O, together with the sum of residues of $e^{xz}F(z)z^{-1/2}$ in the semicircle.

From the assumption: $\left|\frac{F(z)}{z^{1/2}}\right|\to 0$ for $R(z)\leqq c$, $|z|\to\infty$, (b)

we can conclude that the integrals along the integrals along the circular arcs BC, GH tend to 0 as $r\to\infty$. Consequently the integral along the infinite straight line $(c-i\infty,\ c+i\infty)$ is equal to the sum of residues in the half-plane $R(z)<c$ together with the integral along the contour (C) consisting of the negative real axis, from $-\infty$ to $-\varepsilon$, a small circle about the origin with radius ε, and the negative real axis, from $-\varepsilon$ to $-\infty$.

Letting $\varepsilon\to 0$, we have

$$\frac{1}{2\pi i}\int_{(C)}\frac{e^{xz}F(z)}{z^{1/2}}dz=\frac{1}{2\pi i}\int_0^\infty\frac{e^{-x\rho}F(-\rho)}{e^{-i\pi/2}\sqrt{\rho}}d\rho-\frac{1}{2\pi i}\int_0^\infty\frac{e^{-x\rho}F(-\rho)}{e^{i\pi/2}\sqrt{\rho}}d\rho$$

$$=\frac{1}{\pi}\int_0^\infty\frac{e^{-x\rho}F(-\rho)}{\sqrt{\rho}}d\rho \quad (=J(x),\ \text{say}).$$

If $F(-\rho)$ be expansible in the Taylor's series, and $|F^{(n)}(-\rho)|<M_n$ for all $\rho\geqq 0$, then

$$F(-\rho)=a_0-a_1\rho+a_2\rho^2-\cdots+(-1)^n a_n\rho^n+\frac{(-1)^{n+1}\rho^{n+1}}{(n+1)!}F^{(n+1)}(-\theta\rho).$$

Putting this in the integral J, we have

$$J=\frac{1}{\pi}\sum_{k=0}^{n}(-1)^k a_k\int_0^\infty e^{-x\rho}\rho^{k-1/2}d\rho+\frac{1}{\pi}\frac{(-1)^{n+1}}{(n+1)!}\int_0^\infty e^{-r\rho}F^{(n+1)}(-\theta\rho)\rho^{n+1/2}d\rho$$

$$=\frac{1}{\pi}\sum_{k=0}^{n}\frac{(-1)^k a_k}{x^{k+1/2}}\Gamma\left(k+\frac{1}{2}\right)+R_n,$$

where $$|R_n| \leqq \frac{M_{n+1}}{\pi(n+1)!} \frac{\Gamma(n+3/2)}{x^{n+3/2}} .$$

Whence we can conclude that

$$\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k a_k}{x^{k+1/2}}\Gamma(k+1/2)$$

is the asymptotic expansion of J, which is nothing but $S(x)$.

Thus we have

$$h(x)=J(x)+ \text{ sum of residues of } \frac{e^{xz}F(z)}{z^{1/2}}$$

and $$J(x)\sim S(x).$$

For example, when $F(p)=\dfrac{1}{p^2+\omega^2}$,

we get $$h(x)\sim\frac{1}{\omega^{3/2}}\sin\left(\omega x-\frac{\pi}{4}\right)+S(x),$$

$$S(x)=\frac{1}{\omega^2\sqrt{\pi x}}\sum\frac{(-1)^n 1\cdot3\cdot5\ldots(4n-1)}{(2\omega x)^{2n}}.$$

This result coincides with the result obtained by Mr. Carson when $\int_x^\infty\frac{\sin\omega x}{\sqrt{x}}dx$, $\int_x^\infty\frac{\cos\omega x}{\sqrt{x}}dx$ are substituted by the asymptotic expansions.

Next consider the case[1] $F(p)=\dfrac{1}{p-\lambda}$.

In this case, the residue of $e^{xz}F(z)/z^{1/2}$ at the point $z=\lambda$ being $e^{\lambda x}/\sqrt{\lambda}$, we have

$$h(x)\sim\frac{e^{\lambda x}}{\sqrt{\lambda}}+S(x),\quad S(x)=-\frac{1}{\lambda\sqrt{\pi x}}\left\{1-\frac{1}{2\lambda x}+\frac{1\cdot3}{(2\lambda x)^2}-\frac{1\cdot3\cdot5}{(2\lambda x)^3}+\cdots\right\},$$

where λ is assumed not a negative real number.

Thus we have the following theorem.

Theorem. Let $F(z)$ be a meromorphic function whose poles do not lie in the half-plane $R(z)\geqq c>0$, and on the negative real axis, the origin inclusive. Then under the condition (a), (b), *the Carson's solution $h(x)$ of the equation*

$$\frac{F(x)}{x^{1/2}}=\int_0^\infty e^{-xt}h(t)\,dt,\quad \left(F(p)p^{1/2}.\ 1=h(x)\right)$$

is equal to $\dfrac{1}{\pi}\displaystyle\int_0^\infty\frac{e^{-xt}F(-t)}{\sqrt{t}}dt+$ *sum of residues of* $\dfrac{e^{xz}F(z)}{z^{1/2}}$.

The integral here stated has the asymptotic expansion $\dfrac{1}{\sqrt{\pi x}}\sum(-1)^k$

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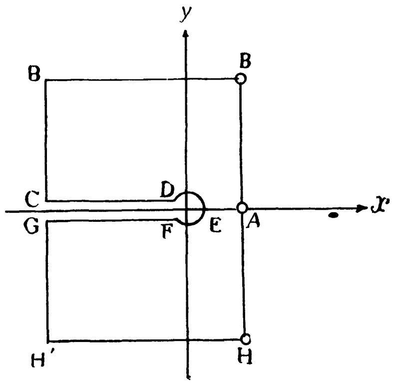
1) Carson, Trans. Amer. M.S., **31** (1929).

$\times 1\cdot 3\cdot 5 \cdots (2n-1)\ (2x)^{-k}$, where $a_n = \dfrac{F^{(n)}(0)}{n!}$, and $|F^{(n)}(-t)|$ is assumed to be bounded for $t \geqq 0$. *This asymptotic series is nothing but the Heaviside's solution.*



3. If we take a rectangle $BB'CH'H$ instead of a semi-circle BCH, then we can proceed as follows.

Let B, B', H', H be represented by $c+iN$, $-M+iN$, $-M-iN$, $c-iN$, then

$$\left|\frac{1}{2\pi i}\int_{B'B}\frac{e^{xz}F(z)}{\sqrt{z}}dz\right| \leqq \frac{1}{2\pi}\int_{-M}^{c} e^{x\sigma}\left|\frac{F(\sigma+iN)}{\sqrt{\sigma+iN}}\right|d\sigma .$$

If $$\left|\frac{F(z)}{\sqrt{z}}\right| \to 0 \text{ uniformly for any finite strip } \alpha \leqq \sigma \leqq \beta , \qquad \text{(c)}$$

when $$|t| \to \infty , \quad (z=\sigma+it) ,$$

then the integral above stated $\to 0$ as $N \to \infty$. The same holds good for the integral along $H'H$.

Again from $$\left|\frac{1}{2\pi i}\int_{-M-i\infty}^{-M+i\infty}\frac{e^{xz}F(z)}{\sqrt{z}}dz\right| \leqq \frac{1}{2\pi}e^{-Mx}\int_{-\infty}^{\infty}\left|\frac{F(-M+it)}{\sqrt{-M+it}}\right|dt ,$$

it follows that the left hand side $\to 0$ as $M \to \infty$, if

$$\int_{-\infty}^{\infty}\left|\frac{F(-\sigma+it)}{\sqrt{-\sigma+it}}\right|dt = o(e^{\varepsilon\sigma}) , \quad \text{for any } \varepsilon > 0 . \qquad \text{(d)}$$

Thus we have the following theorem.

Theorem. When the conditions (a), (c) *and* (d) *instead of* (a) *and* (b) *are satisfied, then we can conclude* $h(x) \sim$ *sum of residues of* $\dfrac{e^{xz}F(z)}{\sqrt{z}} +$ *Heaviside's asymptotic series.*

Added in proof. By the notice of Dr. Izumi I found that the first theorem above stated is contained as a special case in Theorem 1 in a paper by Dr. Bourgin and Duffin, the Heaviside operational calculus, American Journal of Math. **59** (1937). The second theorem is new.