63. *An Abstract Treatment of the Individual Ergodic Theorem.*

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§ 1. *Introduction.* The analytical interests concerning the ergodic theorems may be observed from two viewpoints. Firstly, the ergodic theorems give us means for determining the fixed points und linear operations. Secondly, they concern with the deduction of the stronger convergences of linear operations from the weaker ones. Some entensive literatures on the mean ergodic theorem are more or less guided by these viewpoints. Concerning the individual ergodic theorem, however, we have only small number of literatures. The dominated ergodic theorem due to N. Wiener and M. Fukamiya,[1] and the writer's extensions[2] of Birkhoff-Khintchine's ergodic theorem both constitute examples on the individual ergodic theorem. The purpose of the present note is to extend the idea developed in [I]. The continuity theorems (theorem 1 and its corollary) have interests of their own. Theorem 2 is an abstract form of the individual ergodic theorem. We may deduce from this the individual ergodic theorem for m-parameter abelian group of equi-measure transformations, and more generally that for general semi-group of linear operations (theorem 3). The results in [I] in a somewhat extended form are also deducible from theorem 3.

§ 2. *A continuity theorem and an abstract form of the individual ergodic theorem.* Under abstract (S) space $(A-S)$ we mean a linear space of type-F, which satisfies the following axioms (we donote by $x, y, \dots$ the elements of $(A-S)$, by $\|x\|_S, \|y\|_S, \dots$ their quasi-norms and by λ a real scalar):

(1) a semi-order relation $x>y$ is defined in $(A-S)$, relative to which $(A-S)$ is a linear lattice, viz.:

(1–1) corresponding to any two elements x, y there exist the least upper bound $\sup(x, y)$ and the greatest lower bound $\inf(x, y)$,

(1–2) translations $x\rightarrow x+y$ and homothetic expansions $x\rightarrow\lambda x$ with $\lambda>0$ preserve the semi-ordering,

(1–3) $\sup(x, y)$ and $\inf(x, y)$ are both continuous in x and y in the topology defined by the norm $\|\ \|_S$.

(2) any sequence $\{x_n\}$ bounded from above (below) admits of the least upper bound $\sup\limits_n x_n$ (the greatest lower bound $\inf\limits_n x_n$).

(3) if we write $\operatorname*{Lim}\limits_{n\to\infty} x_n=x$, in case $\overline{\operatorname*{Lim}\limits_{n\to\infty}}\, x_n=\inf\limits_n \sup\limits_{m\geq n} x_m=\underline{\operatorname*{Lim}\limits_{n\to\infty}}\, x_n=\sup\limits_n \inf\limits_{m\geq n} x_m$, then $\operatorname*{Lim}\limits_{n\to\infty} x_n=x$ implies $\lim\limits_{n\to\infty}\|x_n-x\|_S=0$.

(4) $x\geqq y\geqq -x$ implies $\|x\|_S\geqq\|y\|_S$.

1) N. Wiener: Duke Math. J., **5** (1939), 1–18. M. Fukamiya: Tôhoku Math. J., **46** (1939), 150–153. Cf. also K. Yosida and S. Kakutani: Proc. **15** (1939), 165–168.

2) K. Yosida: Jap. J. of Math., **15** (1940), 31–36, to be cited as [I] below.

Let (S) be the class of real-valued measurable functions $x(t)$ on finite or infinite interval (a, b), which are finite almost everywhere. By the quasi-norm $\|x\|_S=\int_b^a \frac{|x(t)|}{1+|x(t)|}\frac{dt}{1+t^2}$ and the semi-order relation $x \geqq y$ $\big(x(t) \geqq y(t)$ almost everywhere$\big)$, (S) constitutes a concrete example of the space $(A-S)$.[1] In this case, $\operatorname{Lim}_{n\to\infty} x_n=x$ means $\lim_{n\to\infty} x_n(t)=x(t)$ almost everywhere. In the sequel, we will write $|x|$ for $\sup(x, 0)$-$\inf(x, 0)$.

After these preliminaries we may prove

Theorem 1.[2] *Let (F) be a space of type-F by the quasi-norm $\|x\|_F$, and consider a sequence $\{T_n\}$ of linear continuous operations on (F) to $(A-S)$. Putting $x_n=T_n\cdot x$ for any $x\in(F)$, we assume that the set $(F)'$ of points $x\in(F)$ for which $\overline{\operatorname{Lim}}_{n\to\infty} x_n$ exist constitutes a set of second category in (F). Then $\overline{\operatorname{Lim}}_{n\to\infty} x_n$ and $\underline{\operatorname{Lim}}_{n\to\infty} x_n$ exist for all $x\in(F)$ and the operation $\widetilde{T}$, $\widetilde{T}\cdot x=\widetilde{x}$, $\widetilde{x}=\overline{\operatorname{Lim}}_{n\to\infty} x_n-\underline{\operatorname{Lim}}_{n\to\infty} x_n$ is a continuous operation on (F) to $(A-S)$.*

Proof. Put $x'_n=\sup_{n\geq m}|x_m|$, $x'=\sup_m |x_m|$ for any $x\in(F)'$, and consider the operations $V_n\cdot x=x'_n$, $V\cdot x=x'$ on $(F)'$ to $(A-S)$. Each V_n is continuous and $\lim_{n\to\infty}\|V_n\cdot x-V\cdot x\|_S=0$. Hence, from $\lim_{n\to\infty}\left\|\frac{1}{k}V_n\cdot x\right\|_S=\left\|\frac{1}{k}V\cdot x\right\|_S$ $(k=1, 2, \dots)$ and $\lim_{n\to\infty}\left\|\frac{1}{k}V\cdot x\right\|_S=0$, we obtain the inclusion $(F)'\leqq\sum_{k=1}^{\infty}G_k$, $G_k=\underset{x\in(F)}{E}\left(\sup_n\left\|\frac{1}{k}V_n\cdot x\right\|_S\leqq\varepsilon\right)$, for any $\varepsilon>0$. G_k is a closed set of (F) by the continuity of V_n. Since $(F)'$ is of second category in the complete metric space (F), there must exist a G_k which contains a sphere of (F). Thus we have a point $x_0\in(F)$ and a constant $\delta>0$ such that $\sup_n\left\|\frac{1}{k}V_n\cdot x\right\|_S\leqq\varepsilon$ for $\|x-x_0\|_F\leqq\delta$. Hence, from $|V_n\cdot(x-x_0)|\leqq|V_n\cdot x|+|V_n\cdot x_0|$ and $V_n\cdot\left(\frac{1}{k}x\right)=\frac{1}{k}V_n\cdot x$, we see that $\lim_{\|x\|_F\to 0}\|V_n\cdot x\|_S=0$ uniformly in n. $V\cdot x$ is thus defined for all $x\in(F)$ and is continuous at $x=0$ with $V\cdot 0=0$. Thus, by $|\widetilde{T}\cdot x|\leqq 2V\cdot x$ and $\|\widetilde{T}\cdot x-\widetilde{T}\cdot y\|_S\leqq\|\widetilde{T}\cdot(x-y)\|_S$, $\widetilde{T}$ is defined and continuous at every $x\in(F)$. Q. E. D.

Corollary. The set $(F)''$ of points $x\in(F)$ for which $\operatorname{Lim}_{n\to\infty} x_n$ exists either coincides with the whole (F) or it constitutes a set of first category in (F).

Proof. Let $(F)''$ be of second category in (F), then $\widetilde{T}\cdot x$ is con-

1) In truth, (S) is a semi-ordered ring. The concrete representation of semi-ordered ring as subring of (S) will be published elsewhere. It is to be noted that the metrical completeness of $(A-S)$ is not necessary in the present note.

2) This is an extension of Banach-Saks' theorem as formulated by S. Mazur and W. Orlicz: Stud. Math., 4 (1933), 152–157.

tinuous on (F) to $(A-S)$. Thus $(F)''=E_x(\widetilde{T}\cdot x=0)$ is a closed set of (F). That $(F)''$ is a linear subspace of (F) is evident. A closed linear subspace which is of second category in a space of type-F must coincide with the whole space. Q. E. D.

Theorem 2. Let the above (F) *be a linear subspace of* $(A-S)$ *such that* $\lim_{n\to\infty}\|x_n-x\|_F=0$ *implies* $\lim_{n\to\infty}\|x_n-x\|_S=0$, *and let* $\{T_n\}$ *be a sequence of linear continuous operations on* (F) *to* (F). *Assume, as above, that* $\overline{Lim}_{n\to\infty}\, x_n$ *exists as point* $\in(A-S)$ *for those* $x\in(F)$ *which constitutes a set of second category in* (F). *If, for a point* $y\in(F)$, *there corresponds* $\bar{y}\in(F)$, *such that* $\lim_{n\to\infty}\|y_n-\bar{y}\|_F=0$, $T_n\cdot\bar{y}=\bar{y}$ $(n=1, 2, \dots)$, $Lim_{n\to\infty}(T_n\cdot y-T_nT_m\cdot y)=0$ $(m=1, 2, \dots)$, *then we must have* $Lim_n\, y_n=\bar{y}$.

Proof. Put $y=\bar{y}+(y-\bar{y})$, then $0\leqq\widetilde{T}\cdot y\leqq\widetilde{T}\cdot(y-\bar{y})$ by $T_n\cdot\bar{y}=\bar{y}$ $(n=1, 2, \dots)$. We have, by theorem 1 and $\lim_{m\to\infty}\|(y-\bar{y})-(y-y_m)\|_F=0$, $\widetilde{T}(y-\bar{y})=0$ from $\widetilde{T}\cdot(y-y_m)=0$ $(m=1, 2, \dots)$. However we have $\widetilde{T}\cdot(y-y_m)=\overline{\mathrm{Lim}}_{n\to\infty}\, T_n(y-T_m\cdot y)-\underline{\mathrm{Lim}}_{n\to\infty}\, T_n(y-T_m\cdot y)=0$ from the assumption. Thus $\mathrm{Lim}_{n\to\infty}\, y_n=z$ exists, and by (3) $\lim_{n\to\infty}\|y_n-z\|_S=0$, which proves $\bar{y}=z$. Hence we have $\mathrm{Lim}_{n\to\infty}\, y_n=\bar{y}$.

§ 3. *The individual ergodic theorem for general semi-groups.* Let $p\geqq1$, then the class of real-valued measurable functions $x(t)$ on (a, b) with $\int_a^b|x(t)|^p dt<+\infty$ constitutes a Banach space (L^p) by the norm $\|x\|=\left(\int_a^b|x(t)|^p dt\right)^{\frac{1}{p}}$.

Let G be a semi-group (multiplicative system) of linear continuous operations $T^{(g)}$ on (L^p) to (L^p) with uniformly bounded norms,

(5) $\|T^{(g)}\|\leqq C$ for all $T^{(g)}\in G$.

Following L. Alaoglu and Garrett Birkhoff,[1] we will call G "ergodic" when it possesses a sequence of measures $\varphi_n(V)$ satisfying i) $\varphi_n(G)=1$ for all n, and ii) given $T^{(g)}$ in G and $\varepsilon>0$, N exists so large that $n\geqq N$ implies $|\varphi_n(VT^{(g)})-\varphi_n(V)|+|\varphi_n(T^{(g)}V)-\varphi_n(V)|<\varepsilon$. We assume that for any $x\in(L^p)$ the integral $T_n\cdot x=\int_G(T^{(g)}\cdot x)\,d\varphi_n$ exists in S. Bochner's or G. Birkhoff's sense $(n=1, 2, \dots)$. We call the point $y\in(L^p)$ "strongly ergodic" if we have

(6) $\lim_{n\to\infty}|y_{n,m}(t)-y_n(t)|=0$ almost everywhere for $m=1, 2, \dots$,

where $y_n=T_n\cdot y\,,\quad y_{n,m}=T_nT_m\cdot y\,.$

Then we have the

Theorem 3. We assume that

(7) $\overline{lim}_{n\to\infty}|x_n(t)|<\infty$ *almost everywhere for those* x *which constitutes a set of second category in* (L^p).

1) L. Alaoglu and Garrett Birkhoff: Ann. of Math., **41** (1940), 293–309.

Let a strongly ergodic point $y \in (L^p)$ *satisfy the condition.*

(8) $\{y_n\}$ *contains a subsequence weakly convergent to a point* $\bar{y} \in (L^p)$.[1)]

Then $\lim_{n\to\infty} y_n(t) = \bar{y}(t)$ *almost everywhere,* $\lim_{n\to\infty} \|y_n - \bar{y}\| = 0$ *and* $T^{(g)}\bar{y} = \bar{y}$ *for all* $T^{(g)} \in G$.

Proof. From (5) we have $\|T_n\| \leqq C$ $(n=1, 2, \dots)$. Hence, by (8) and the ergodicity of G, we see[2)] that $\lim_{n\to\infty} \|y_n - \bar{y}\| = 0$ and $T^{(g)} \cdot \bar{y} = \bar{y}$ for all $T^{(g)} \in G$. Hence, by (6), (7) and theorem 2, we obtain the theorem. Q. E. D.

§ 4. *Application of theorem 3.* Let $T^{(g)}$ be defined by equi-measure transformation $P^{(g)}$: $x^{(g)} = T^{(g)} \cdot x$, $x^{(g)}(t) = x(P^{(g)} \cdot t)$ for $x \in (L^p)$, and let $G = \{T^{(g)}\}$ be an m-parameter abelian group of such $T^{(g)}$. Then each $T^{(g)}$ satisfies (5) with $C=1$ as linear operation on (L^1) and on (L^2). By putting $\varphi_n(V)$ = the proportion of the cube $0 \leqq \lambda_1 \leqq n$, $0 \leqq \lambda_2 \leqq n$, $\dots$, $0 \leqq \lambda_m \leqq n$, to its portion occupied by V, we see that G is ergodic. We have, moreover, (7) for $p=1$ and for $p=2$, by the dominated ergodic theorem of N. Wiener.[3)] That (6) is satisfied for all $y \in (L^2)$ may be proved in the following manner. For any $y \in (L^2)$ and k we have $\|y_{n,k} - y_n\| \leqq C_k \frac{1}{n} \|y\|$ $(n=1, 2, \dots)$, by the definition of $\varphi_n(V)$. Hence, if we put $E(\varepsilon, n, k) = \underset{t}{E}(|y_{n,k}(t) - y_n(t)| \geqq \varepsilon)$, we have mes $\big(E(\varepsilon, n, k)\big) \leqq \frac{C_k^2}{\varepsilon^2} \cdot \frac{1}{n^2} \|y\|^2$, proving (6) by $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.[4)] Since (L^2) is locally weakly compact, we have (8) for any $y \in (L^2)$. Thus, by theorem 3, $\lim_{n\to\infty} y_n(t)$ exists almost everywhere for all $y \in (L^2)$. Since (L^2) is strongly dense in (L^1) by the norm of (L^1), we have also the existence almost everywhere of $\lim_{n\to\infty} y_n(t)$ for all $y \in (L^1)$, by theorem 1.

Thus we have proved N. Wiener's *m-parameter individual ergodic theorem.*[5)]

Next let G be the semi-group generated by the iterations $\{T^m\}$ of a linear operation T on (L^p) to (L^p), and put $T_n = \frac{1}{n} \sum_{m=1}^{n} T^m$ $(n=1, 2, \dots)$. Then, as above, we may prove by theorem 1 and 3 the following individual ergodic theorems. Since, their proofs are similar as those in [I], we omit them. However, it is to be noted that the results are somewhat more general than in [I], for we here deal with (L^p) on finite or infinite interval (a, b).

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1) If $p > 1$, (8) is superflous, since (L^p) with $p > 1$ is locally weakly compact.

2) L. Alaoglu and G. Birkhoff: loc. cit. It is to be noted that we may obtain the same result by the arguments employed in the proof of the mean ergodic theorem of F. Riesz, S. Kakutani and the present author.

3) N. Wiener: loc. cit. Though we here are concerned with (L^p) on finite or infinite interval (a, b), we may obtain (7) by Wiener's argument.

4) This device I owe to M. Fukamiya.

5) N. Wiener: loc. cit. Wiener's proof applies to (L^1) on finite interval (a, b) only; the mean ergodic theorem for (L^1) on infinite interval does not in general hold good. Our proof, however, applies to (L^1) on finite as well as on infinite interval (a, b).

Theorem 4. *Let T be a linear continuous operation (L^1) to (L^1) and put $x^{(n)}=T^n\cdot x$ for any $x\in(L^1)$. We assume that*

(9) $\|T^n\|\leqq$ *a constant* C $(n=1,2,\ldots)$,

(10) $\overline{\lim_{n\to\infty}}\left|\frac{1}{n}\sum_{m=1}^{n}x^{(m)}(t)\right|<\infty$ *almost everywhere for those x which constitute a set of second category in (L^1).*

If for an element $y\in(L^1)$

$\lim_{n\to\infty}\frac{y^{(n)}(t)}{n}=0$ *almost everywhere,*

the sequence $\left\{\frac{1}{n}\sum_{m=1}^{n}y^{(m)}\right\}$ *contains a subsequence weakly convergent to an element $\bar{y}\in(L^1)$,*

then we have

(11) $\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{m=1}^{n}y^{(m)}-\bar{y}\right\|=0$, $\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}y^{(m)}(t)=\bar{y}(t)$ *almgst everywhere.*

Theorem 5. *Let $p>1$, and let T be a linear continuous operation on (L^p) to (L^p) satisfying* (9) *and* (10) *in (L^p). Then we have* (11) *for all $y\in(L^p)$.*

Theorem 6. *Let T be a linear operation on (L^p) to (L^p), $p\geqq1$. We assume that*

for any $x\in(L^p)$ there corresponds a $X\in(L^p)$ such that

$|x^{(n)}(t)|\leqq X(t)$ *almost everywhere* $(n=1,2,\ldots)$.

Then (11) *hold good for any $y\in(L^p)$.*[1]

Remark. The application of theorem 1, 2 and 3 is not exhausted by the above theorems. We may obtain the individual ergodic theorem in the space of abstractly-valued functions. We may also prove and extend B. Jessen's theorem of approximation of Lebesgue integral by Riemann sums.[2] We will here not enter into the details.

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1) This is more precise than the results of Garrett Birkhoff: Proc. Nat. Acad. Sc., **24** (1938), 154-159, F. Riesz: Proc. London Math. Soc., **13** (1938), 274-278 and S. Kakutani: Proc. **15** (1939), 121-123.

2) B. Jessen: Ann. of Math., **35** (1934), 248-251.